Problems Chapter 5

Quantum Mechanics
K. Konishi, G. Paffuti

Problem 1

A particle of spin J=1 and unknown parity decays at rest in two identical spin 1/2 particles.

1. Compute the orbital angular momentum of the final particles and the total spin.
2. Determine the parity of the decaying particle, assuming that in the decay parity is conserved.
3. Let us suppose that the initial particle is in the state $|J, J_z\rangle = |1, 0\rangle$. Write explicitly the final state with an unknown radial function, but with explicit use of spherical harmonics and spin states.

Let us suppose now that one measures the spin projections of the final particles with Stern-Gerlach type apparatus, for two particles emitted in the directions (θ,φ) and (π-θ, φ+π).

- **a.** For fixed values of θ, φ, write the normalized spin wave function of the final state.
- **b.** Compute the probability that a particle is emitted in the direction θ, φ with s_z= +1/2 .
- **c.** Compute the probability that a simultaneous measurement of $(s_y(1),\ s_y(2))$ gives (1/2, 1/2).

Solution

1

The possible values for S, L compatible with J = 1 are

$$S = 1,\ 0;\quad L = 0,\ 1,\ 2\,. \tag{1.1}$$

The final particles are fermions, the final state must be completely antisymmetric, the only possibility left is S=1, L=1, symmetric in spin and antisymmetric in orbital variables.

2

The parity of the final state is

$$P = (-1)^L; \tag{1.2}$$

then if parity is conserved P = -1.

3

Using the Clebsch - Gordan coefficients to sum L and S the wave function can be written as

$$\psi = R[r]\frac{1}{\sqrt{2}}\left(Y_{11}[\theta,\varphi]\,|1,-1\rangle - Y_{1-1}[\theta,\varphi]\,|1,+1\rangle\right) =$$

$$= R[r]\frac{-i}{\sqrt{2}}\sqrt{\frac{3}{8\pi}}\left\{\mathrm{Sin}[\theta]\,e^{i\varphi}\,|\downarrow\downarrow\rangle + \mathrm{Sin}[\theta]\,e^{-i\varphi}\,|\uparrow\uparrow\rangle\right\}.$$

a

The normalized spin state is

$$\psi_{spin} = \frac{1}{\sqrt{2}}\left(e^{i\varphi}\,|\downarrow\downarrow\rangle + e^{-i\varphi}\,|\uparrow\uparrow\rangle\right). \tag{1.3}$$

b

The probability is 1/2.

c

The eigenstate with $(s_y(1),\ s_y(2))$ = (1/2,1/2) is

$$\psi_1 = \frac{1}{2}\left(|\uparrow\rangle + i\,|\downarrow\rangle\right)\left(|\uparrow\rangle + i\,|\downarrow\rangle\right) = \frac{1}{2}\left(|\uparrow\uparrow\rangle + i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right);$$

The requested probability is

$$|\langle \psi_1 | \psi_{spin}\rangle|^2 = \frac{\mathrm{Sin}[\varphi]^2}{2}.$$

Problem 2

A deuteron d is a nucleus with charge +1, composed of a proton (p) and a neutron (n). The deuteron has spin 1 and parity +. A negative pion π^-, with charge -1 and spin 0, can be bound to the deuteron to form a sort of "deuterium atom". Let us suppose that this system is formed in the lowest Bohr orbit.

1. Compute the ratio between the Bohr radius of this system and the standard Bohr radius, and compute the binding energy of the system. Some masses needed for the computation are listed below:

$$M_{\pi^-} \simeq 139.6\,\mathrm{MeV}/c^2;\; M_{e^-} \simeq 0.51\,\mathrm{MeV}/c^2;\; M_d \simeq 1875.6\,\mathrm{MeV}/c^2;\; M_p \simeq 938.3\,\mathrm{MeV}/c^2.$$

2. The bound system described above decays with the reaction $\pi^- + d \to n + n$. Both angular momentum and parity are conserved in the decay. Discuss if one can determine the intrinsic parity of the π^- from these data.
3. Compute the angular distribution of neutrons in the final state knowing that in the initial state $J_z = 0$.
4. Explain why the hydrogen atom does not decay via a somewhat analogous process, $e^- + p \to n + \nu$, where ν is the (electron) neutrino.

Solution

1

The reduced mass of the system is

$$\mu = \frac{M_{\pi^-} M_d}{M_{\pi^-} + M_d} \simeq 129.9\,\mathrm{MeV}/c^2.$$

The ratio of the Bohr radius with respect the usual one is

$$\frac{r'_B}{r_B} \simeq \frac{M_e}{\mu} = \frac{0.51}{129.9} = 0.0039.$$

The binding energy

$$\frac{e^2}{2\,r'_B} = \frac{e^2}{2\,r_B}\frac{r_B}{r'_B} = \frac{13.6}{0.0039}\,\mathrm{eV} \simeq 3.49\,\mathrm{KeV}.$$

2

The orbital angular momentum for the bound state is zero, the pion's spin is zero, then the initial angular momentum is one, the spin of the deuteron.

The states allowed by Fermi statistics for the final neutrons are

a. S=0 ; L = 0, 2, 4...
b. S=1 ; L = 1, 3, 5

The only combination compatible with J = 1 is (S=1, L = 1). The final state then has parity -1. The parity of the initial state is

$$P_{in} = P_\pi\,(-1)^\ell = P_\pi$$

then parity conservation imply $P_\pi = -1$. This was indeed the method used to determine experimentally pion's intrinsic parity.

3

The bound state is in the angular momentum state $|J, J_z\rangle = |1, 0\rangle$. The same state must describe final particles. Decomposing the final state in terms of orbital and spin states

$$\psi = \frac{1}{\sqrt{2}}\left(Y_{11}[\theta, \varphi]\,|1, -1\rangle - Y_{1-1}[\theta, \varphi]\,|1, +1\rangle\right).$$

The angular distribution of the decay' s products is then

$$P\,d\Omega = \frac{3}{8\pi}\mathrm{Sin}[\theta]^2\,d\Omega. \qquad \int P\,d\Omega = 1.$$

More explicitly, the final state is

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(| 1, +1 \rangle_L \; | 1, -1 \rangle_S - | 1, -1 \rangle_L \; | 1, +1 \rangle \right);$$

the probability to detect a particle in $d\Omega$ without, independently of the spin, is

$$P = |\langle 1, 1 |_S \langle \Theta, \varphi | \psi \rangle|^2 + |\langle 1, -1 |_S \langle \Theta, \varphi | \psi \rangle|^2 = \frac{1}{2} \left(|Y_{11}[\Theta, \varphi]|^2 + |Y_{1,-1}[\Theta, \varphi]|^2 \right) = \frac{3}{8\pi} \mathrm{Sin}[\Theta]^2.$$

■ 4

The process $e^- + p \to n + \nu$ is perfectly allowed from the point of view of the quantum numbers: in fact, the neutron decays as $n \to p + e^- + \bar{\nu}$ where $\bar{\nu}$ is the anti-netrino (beta decay). But as this very fact shows, the mass of the neutron is larger than the sum of those of the proton, electron and neutrino. As the mass of the hydrogen atom is smaller than the sum of the proton mass and electron mass (by Borh's binding energy) this process is forbidden by (relativistic) energy conservation.

Problem 3

Write the explicit form of the completeness relation for a system of two free identical particles.

Solution

For a single particle the completeness relation is formally written as

$$1 = \sum_i |i\rangle\langle i| .$$

Let us now consider a system of N particles. The completeness take the form

$$1 = \sum_{\lambda_1 ..}' |\psi[\lambda_1, \ldots]\rangle\langle\psi[\lambda_1, \ldots]| . \quad (3.1)$$

We use the symbol ψ[] to denote the symmetric ket.

The sum runs on inequivalent permutations (i.e. those corresponding to effectively different states). For the continuous spectrum this would imply some complicated excluded domains in the integrals. It is simpler to sum on all configurations. There are N! /(n_1 ! n_2 ! ..) of such permutations, then it is simpler to write

$$1 = \frac{n_1 ! \, n_2 ! \, ..}{N !} \sum_{\lambda_1 ..} |\psi[\lambda_1, \ldots]\rangle\langle\psi[\lambda_1, \ldots]| . \quad (3.2)$$

In the following we will be interested in continuum states, in this case the relative probability for two or more particles to be in the same cell of phase space is negligible and we can assume n_i=1 in all formulas.

In this approximation we can write, for two particles (the formulas below are valid both for bosons and fermions)

$$1 = \frac{1}{2!} \int_{k_1 k_2} |\psi[k_1, k_2]\rangle\langle\psi[k_1, k_2]| \quad (3.3)$$

Let us check this relation. For bosons or fermions we have

$$|\psi[k_1, k_2]\rangle = \frac{1}{\sqrt{2}} (|k_1, k_2\rangle \pm |k_2, k_1\rangle);$$

The scalar product between two states is

$$\langle\psi[q_1, q_2] | \psi[p_1, p_2]\rangle = \left(\delta_{q_1 p_1} \delta_{q_2 p_2} \pm \delta_{q_1 p_2} \delta_{q_2 p_1}\right).$$

Using (3) we get the same result:

$$\frac{1}{2} \int_{k_1 k_2} \langle\psi[q_1, q_2] | \psi[k_1, k_2]\rangle\langle\psi[k_1, k_2] | \psi[p_1, p_2]\rangle = \frac{1}{2} \left(\delta_{q_1 k_1} \delta_{q_2 k_2} \pm \delta_{q_1 k_2} \delta_{q_2 k_1}\right) \left(\delta_{k_1 p_1} \delta_{k_2 p_2} \pm \delta_{k_1 p_2} \delta_{k_2 p_1}\right) = \left(\delta_{q_1 p_1} \delta_{q_2 p_2} \pm \delta_{q_1 p_2} \delta_{q_2 p_1}\right)$$

Problem 4

Write the Hamiltonian and the Heisenberg equations for a system of particles interacting with a potential $U[x_1, x_2]$ *in the second quantization formalism.*

Solution

Hamiltonian

The kinetic term of the Hamiltonian in Fock representation is

$$H_0 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, a^\dagger[\mathbf{k}]\, \frac{\hbar^2\,\mathbf{k}^2}{2\,m}\, a[\mathbf{k}]. \tag{4.1}$$

The field operator Φ[x] is defined by

$$\Phi[x] = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, a[\mathbf{k}]\, e^{i\,\mathbf{k}\,\mathbf{x}}. \tag{4.2}$$

We get easily

$$H_0 = \int d^3x\, \Phi^\dagger[\mathbf{x}] \left(-\hbar^2 \frac{\nabla^2}{2\,m}\right) \Phi[\mathbf{x}] \tag{4.3}$$

In fact, by using (2)

$$\int d^3x\, \Phi^\dagger[\mathbf{x}] \left(-\hbar^2 \frac{\nabla^2}{2\,m}\right) \Phi[\mathbf{x}] = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int d^3x\; a^\dagger[\mathbf{q}]\, e^{-i\,\mathbf{q}\,\mathbf{x}}\, \frac{\hbar^2\,\mathbf{k}^2}{2\,m}\, a[\mathbf{k}]\, e^{i\,\mathbf{k}\,\mathbf{x}}$$

The space integral gives $(2\pi)^3\,\delta^3(\mathbf{k}-\mathrm{q})$ and the result follows.
In the Fock representation the interaction term is a two particle operator and is given by

$$H_I = \frac{1}{2}\int_{k_1\,k_2\,k_3\,k_4} a^\dagger[\mathbf{k}_1]\, a^\dagger[\mathbf{k}_2]\, \langle\, \mathbf{k}_1, \mathbf{k}_2 \mid U \mid \mathbf{k}_3, \mathbf{k}_4 \,\rangle\, a[\mathbf{k}_3]\, a[\mathbf{k}_4]. \tag{4.4}$$

The matrix element is

$$\langle\, \mathbf{k}_1, \mathbf{k}_2 \mid U \mid \mathbf{k}_3, \mathbf{k}_4 \,\rangle = \int d^3\mathbf{x}_1\, d^3\mathbf{x}_2\, e^{-i\,(\mathbf{k}_1\,\mathbf{x}_1 + \mathbf{k}_2\,\mathbf{x}_2)}\, U[x_1, x_2]\, e^{i\,(\mathbf{k}_3\,\mathbf{x}_1 + \mathbf{k}_4\,\mathbf{x}_2)}.$$

Inserting this expression into (4) we get

$$H_I = \frac{1}{2}\int d^3\mathbf{x}_1\, d^3\mathbf{x}_2\, \Phi^\dagger[\mathbf{x}_1]\, \Phi^\dagger[\mathbf{x}_2]\, U[x_1, x_2]\, \Phi[\mathbf{x}_1]\, \Phi[\mathbf{x}_2]. \tag{4.5}$$

In a similar way it is easy to show that a possible external field $V^{(E)}[x]$ gives

$$V = \int d^3\mathbf{x}\, \Phi^\dagger[\mathbf{x}]\, \Phi[\mathbf{x}]\, V^{(E)}[\mathbf{x}]. \tag{4.6}$$

The Hamiltonian is given by

$$\begin{aligned} H = {} & \int d^3x\, \Phi^\dagger[\mathbf{x}] \left(-\hbar^2 \frac{\nabla^2}{2\,m}\right) \Phi[\mathbf{x}] + \\ & \frac{1}{2}\int d^3\mathbf{x}_1\, d^3\mathbf{x}_2\, \Phi^\dagger[\mathbf{x}_1]\, \Phi^\dagger[\mathbf{x}_2]\, U[x_1, x_2]\, \Phi[\mathbf{x}_1]\, \Phi[\mathbf{x}_2] + \int d^3\mathbf{x}\, \Phi^\dagger[\mathbf{x}]\, \Phi[\mathbf{x}]\, V^{(E)}[\mathbf{x}]. \end{aligned} \tag{4.7}$$

The Heisenberg equations of motion

We can now consider the Heisenberg representation for the fields, Φ[t,**x**].

$$\Phi[t, x] = \mathrm{Exp}\left[i\,\frac{H}{\hbar}\,t\right] \Phi[x]\, \mathrm{Exp}\left[-i\,\frac{H}{\hbar}\,t\right]. \tag{4.8}$$

From

$$\left[a[\mathbf{k}], a^\dagger[\mathbf{q}]\right] = (2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{q}), \tag{4.9}$$

it follows that at **equal times**

$$[\Phi[t, \mathbf{x}], \Phi[t, \mathbf{y}]] = \left[\Phi^\dagger[t, \mathbf{x}], \Phi^\dagger[t, \mathbf{y}]\right] = 0; \quad \left[\Phi[t, \mathbf{x}], \Phi^\dagger[t, \mathbf{y}]\right] = \delta^3(\mathbf{x} - \mathbf{y}). \tag{4.10}$$

Using these commutations relation a straightforward calculation gives

$$i\hbar \frac{d}{dt} \Phi[t, \mathbf{x}] = [\Phi[t, \mathbf{x}], H] = -\frac{\hbar^2}{2m} \Delta \Phi[t, \mathbf{x}] + V^{(E)}[\mathbf{x}] \Phi[\mathbf{x}] + \int d^3\mathbf{y} \, |\Phi[t, \mathbf{y}]|^2 U[x, y] \Phi[\mathbf{x}]. \tag{4.11}$$

This is the analogue of the Maxwell equations for the Φ field.

Problem 5

Consider a simplified version of the Young experiment limiting the dynamics to two single "modes" of the electromagnetic field, describing a photon passing from the slit 1 or 2 respectively. Use a Fock formalism to describe the photon's beam and compute the probability to measure k times a photon through slit 1 in a single photon experiment repeated n times. Show that the same probability is obtained by measuring k photons in a single experiment with a beam of n photons. Describe in this model the interference.

Solution

Measure of k photons

We denote by $|1\rangle$ and $|2\rangle$ the two photon states corresponding to a passage through slit 1 and 2 in Young's experiment respectively. We will consider only one polarization state. In Fock representation we can write

$$|1\rangle = a_1^\dagger |0\rangle; \quad |2\rangle = a_2^\dagger |0\rangle.$$

The photon's state before the screen is

$$|\gamma\rangle = \frac{1}{\sqrt{2}} \left(a_1^\dagger + a_2^\dagger\right) |0\rangle. \tag{5.1}$$

The measure of photon through the slit 1 and 2 is given by $a_1^\dagger a_1$ and $a_2^\dagger a_2$.
With single photon experiments the probability to pass through each slit is, as expected:

$$\langle \gamma | a_1^\dagger a_1 | \gamma \rangle = \frac{1}{2}; \quad \langle \gamma | a_2^\dagger a_2 | \gamma \rangle = \frac{1}{2}. \tag{5.2}$$

Consider now the experiment repeated n times. The probability to find k times a photon through 1 is given by the binomial distribution

$$P_k = \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \binom{n}{k} = \left(\frac{1}{2}\right)^n \binom{n}{k}. \tag{5.3}$$

Let us consider now an n - photons state. The operators

$$a^\dagger = \frac{1}{\sqrt{2}} \left(a_1^\dagger + a_2^\dagger\right); \quad a = \frac{1}{\sqrt{2}} (a_1 + a_2);$$

satisfy the usual annihilation - creation operators algebra

$$\left[a, a^\dagger\right] = 1,$$

then the correctly normalized n - photon state (before the screen) is given by

$$|n\gamma\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle. \tag{5.4}$$

Expanding $a^\dagger$ we find

$$|n\gamma\rangle = \frac{1}{\sqrt{n!}} \sum_{k=0}^{n} \frac{1}{2^{n/2}} \binom{n}{k} \left(a_1^\dagger\right)^k \left(a_2^\dagger\right)^{n-k} |0\rangle = \sum_{k=0}^{n} \sqrt{\frac{k!\,(n-k)!}{n!}} \frac{1}{2^{n/2}} \binom{n}{k} |(k)_1, (n-k)_2\rangle =$$

$$= \sum_{k=0}^{n} \frac{1}{2^{n/2}} \sqrt{\binom{n}{k}} |(k)_1, (n-k)_2\rangle.$$

From general rules of Quantum Mechanics the probability to measure k photons in slit 1 (and n-k in slit 2) is

$$P'_k = \left| \frac{1}{2^{n/2}} \sqrt{\binom{n}{k}} \right|^2 = \frac{1}{2^n} \binom{n}{k} = P_n .$$

- **Interference**

We can take as approximate value of the electric field in Fock representation

$$E[t, x] = a_1 e^{-i\,(\omega t - k r_1)} + a_2 e^{-i\,(\omega t - k r_2)} = e^{i\,\alpha} \left(a_1 + a_2 e^{i\,\varphi}\right). \tag{5.5}$$

k r_1 and k $r_2$are the optical path and the phase difference φ is

$$\varphi = k\,(r_2 - r_1). \tag{5.6}$$

The measured intensity is proportional to

$$I \propto \langle \gamma \mid E^\dagger E \mid \gamma \rangle .$$

Using

$$E \mid \gamma\rangle \simeq \left(a_1 + a_2 e^{i\,\varphi}\right) \frac{1}{\sqrt{2}} \left(a_1^\dagger + a_2^\dagger\right) \mid 0 \rangle = \frac{1}{\sqrt{2}} \left(1 + e^{i\,\varphi}\right) \mid 0 \rangle$$

we get the usual form of the interference

$$I \propto \frac{1}{2} \left| 1 + e^{i\,\varphi} \right|^2 = (1 + \mathrm{Cos}[\varphi]) .$$