Problems Chapter 7

Quantum Mechanics
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Problem 1

Solve the Heisenberg equations of motion for a free particle and for a particle in an external uniform field.

Solution

The Heisenberg equations for

$$H = \frac{p^2}{2m},$$

read

$$\frac{d}{dt} q_H[t] = -\frac{i}{\hbar}[q_H[t], H] = \frac{p_H[t]}{m}; \quad \frac{d}{dt} p_H[t] = -\frac{i}{\hbar}[p_H[t], H] = 0 . \tag{1.1}$$

From the second equation we get $p_H[t] = p_H[0] = p$. The solution of the first equation is

$$q_H[t] = q + \frac{p}{m} t . \tag{1.2}$$

For a uniform external field

$$H = \frac{p^2}{2m} - F q .$$

The Heisenberg equations are

$$\frac{d}{dt} q_H[t] = -\frac{i}{\hbar}[q_H[t], H] = \frac{p_H[t]}{m}; \quad \frac{d}{dt} p_H[t] = -\frac{i}{\hbar}[p_H[t], H] = F . \tag{1.3}$$

The second equation has solution

$$p_H[t] = p + F t . \tag{1.4}$$

By substitution in the first equation and time integration we find

$$q_H[t] = q + \frac{p}{m} t + \frac{1}{2}\frac{F}{m} t^2 . \tag{1.5}$$

Problem 2

Solve the Heisenberg equations of motion for a harmonic oscillator in a uniform constant external field. Generalize the solution for a uniform time dependent force F[t].

Solution

The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 - F q . \tag{2.1}$$

The Heisenberg equations of motion

$$\frac{d}{dt} q_H[t] = -\frac{i}{\hbar}[q_H[t], H] = \frac{p_H[t]}{m}; \quad \frac{d}{dt} p_H[t] = -\frac{i}{\hbar}[p_H[t], H] = -m\,\omega^2 q + F. \tag{2.2}$$

The change of variables

$$q = Q + \frac{F}{m\,\omega^2}; \quad p = P;$$

is a canonical transformation which leaves commutators invariant and transform the equations in

$$\frac{d}{dt} Q_H[t] = \frac{P_H[t]}{m}; \quad \frac{d}{dt} P_H[t] = -m\,\omega^2 Q_H[t]. \tag{2.3}$$

These are the usual harmonic oscillator equation, the solution has been found in the text and is easily checked to be

$$Q_H[t] = Q[0]\,\mathrm{Cos}[\omega\, t] + \frac{P[0]}{m\,\omega}\mathrm{Sin}[\omega\, t]; \quad P_H[t] = P[0]\,\mathrm{Cos}[\omega\, t] - m\,\omega\, Q[0]\,\mathrm{Sin}[\omega\, t]. \tag{2.4}$$

In terms of p and q :

$$\begin{aligned} q_H[t] &= \frac{F}{m\,\omega^2} + \left(q - \frac{F}{m\,\omega^2}\right)\mathrm{Cos}[\omega\, t] + \frac{p}{m\,\omega}\mathrm{Sin}[\omega\, t]; \\ P_H[t] &= p\,\mathrm{Cos}[\omega\, t] - m\,\omega\left(q - \frac{F}{m\,\omega^2}\right)\mathrm{Sin}[\omega\, t]. \end{aligned} \tag{2.5}$$

The procedure can be generalized to the case of a time dependent force. The equations are the same:

Let us make the canonical (unitary) transformation

$$q = Q + f_q[t]; \quad p = P + f_p[t];$$

The equations of motion become

$$\frac{d}{dt} Q_H[t] = \frac{P_H[t]}{m} + \left(\frac{d}{dt} f_q - \frac{1}{m} f_p\right); \tag{2.6}$$

$$\frac{d}{dt} P_H[t] = -m\,\omega^2 Q_H[t] + \left(\frac{d}{dt} f_p + m\,\omega^2 f_q - F[t]\right) \tag{2.7}$$

If we choose $f_p$and f_q as a particular solution of the classical equations of motion, the equations for P and Q reduce to the usual harmonic oscillator equations, then we can write at once the solution for q and p:

$$\begin{aligned} q_H[t] &= f_q[t] + (q - f_q[t])\,\mathrm{Cos}[\omega\, t] + \frac{(p - f_p[t])}{m\,\omega}\mathrm{Sin}[\omega\, t]; \\ P_H[t] &= f_p[t] + (p - f_p[t])\,\mathrm{Cos}[\omega\, t] - m\,\omega\,(q - f_q[t])\,\mathrm{Sin}[\omega\, t]. \end{aligned} \tag{2.8}$$

A particular solution of the classical equations of motion can be easily find using the method of "variation of constants". Write

$$f_q[t] = \alpha[t]\,\mathrm{Cos}[\omega\, t] + \frac{\beta[t]}{m\,\omega}\mathrm{Sin}[\omega\, t]; \quad f_p[t] = \beta[t]\,\mathrm{Cos}[\omega\, t] - m\,\omega\,\alpha[t]\,\mathrm{Sin}[\omega\, t];$$

Substitution in the equation of motion give

$$\frac{d\alpha}{dt} = -\frac{d\beta}{dt}\frac{\mathrm{Sin}[\omega\, t]}{m\,\omega}; \quad \frac{d\beta}{dt}\mathrm{Cos}[\omega\, t] - m\,\omega\frac{d\alpha}{dt}\mathrm{Sin}[\omega\, t] = F[t].$$

Substituting in the second equation the first one get, as a particular solution (with $\beta[0] = 0$):

$$\beta[t] = \int_0^t F[\tau]\,\mathrm{Cos}[\omega\,\tau]\, d\tau$$

and from the first equation:

$$\alpha[t] = -\frac{1}{m\,\omega}\int_0^t F[\tau]\,\mathrm{Sin}[\omega\,\tau]\, d\tau.$$

Problem 3

Suppose that the system described by the wave function ψ_S *at the instant* $t = 0$ *is an eigenstate of the operator* f, *with eigenvalue,* f_0. *Show that the wave function at time* t *is an eigenstate of the Heisenberg operator* $f_H[-t]$, *with the same eigenvalue.*

Solution

We have

$$f_H[-t]\,\psi[t] = e^{-i H t} f[0]\, e^{i H t} e^{-i H t} \psi[0] = e^{-i H t} f[0]\, \psi[0] = f_0\, e^{-i H t} \psi[0] = f_0\, \psi[t]. \qquad (3.1)$$