Electromagnetic Duality and Solitons in Supersymmetric Gauge Theories

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Contents

1. Introduction

As recent developments show, supersymmetric gauge theories reveal surprisingly deep features of the nonperturbative dynamics of non-Abelian gauge theories, notably the quantum behavior of solitons, such as magnetic monopoles and vortices. Some of these findings might be relevant to the understanding of confinement in QCD. In these lectures an elementary introduction to this field of research will be given, with an emphasis on

(i) Characteristic features of supersymmetric gauge theories;

(ii) Introductory account of anomalies;

(iii) Instantons;

(iv) Solitons and non-abelian electromagnetic duality;

(v) Seiberg-Witten exact soluition of $\mathcal{N}=2$ gauge theories;

(vi) CDSW, generalized Konishi anomalies and exact results on $\mathcal{N}=1$ gauge theories.

2. Two-component spinors

Let us review first the notation and conventions on spinors, following Wess and Bagger[1]. Note however that our convention for the metric tensor is that of Bjorken-Drell:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.1}$$

Two-component spinors are objects which transform according to the representations $\psi \sim (1/2, 0)$, $\bar{\psi} \sim (0, 1/2)$ of the Lorentz group $L_+^\uparrow = SL(2,\mathbb{C}) \sim SU(2) \times SU(2)$. They trasform as

$$\psi_\alpha \to \psi'_\alpha = M_\alpha^{\ \beta}\,\psi_\beta; \qquad \psi^\alpha \to \psi^{\alpha\,\prime} = (M^{-1})_\beta^{\ \alpha}\,\psi^\beta; \qquad \psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta, \tag{2.2}$$

$$\bar{\psi}_{\dot\alpha} \to \bar{\psi}'_{\dot\alpha} = M^{*\,\dot\beta}_{\dot\alpha}\,\bar{\psi}_{\dot\beta}; \qquad \bar{\psi}^{\dot\alpha} \to \bar{\psi}^{\dot\alpha\prime} = (M^*)^{-1\,\dot\alpha}_{\ \dot\beta}\,\bar{\psi}^{\dot\beta}; \qquad \bar{\psi}^{\dot\alpha} \equiv \epsilon^{\dot\alpha\dot\beta}\bar{\psi}_{\dot\beta}, \tag{2.3}$$

$$\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1; \tag{2.4}$$

under Lorentz transformation. The M matrix ($\det M = 1$) is of the form $M = e^{\frac{i}{2}\phi\cdot\sigma}$ for rotations and $M = e^{\frac{1}{2}\omega\cdot\sigma}$ for boosts. Note that

$$\bar{\psi}_1 \equiv {\psi_1}^\dagger, \qquad \bar{\psi}_2 \equiv {\psi_2}^\dagger. \tag{2.5}$$

The vectors

$$P \equiv p_\mu \sigma^\mu = \begin{pmatrix} -p^0 + p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & -p^0 - p^3 \end{pmatrix}; \qquad \sigma^\mu = (-1, \sigma^i), \quad \bar{\sigma}^\mu = (-1, -\sigma^i), \tag{2.6}$$

where the σ^i's are Pauli matrices, transform as

$$P \to P' = M\, P\, M^\dagger \tag{2.7}$$

under which

$$\det P = (p^0)^2 - \mathbf{p}^2 \tag{2.8}$$

is invariant as $\det M = 1$. In fact, Lorentz vectors transform in the same way as the product of two spinors,

$$\psi_\alpha\, \bar{\psi}_{\dot{\alpha}} \sim k_\mu\, \sigma^\mu_{\alpha\dot{\alpha}} \sim (1/2, 1/2) \tag{2.9}$$

while

$$\psi^\alpha\, \psi_\alpha, \quad \bar{\psi}_{\dot{\alpha}}\, \bar{\psi}^{\dot{\alpha}}, \quad \psi^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \bar{\psi}^{\dot{\alpha}} \tag{2.10}$$

summed over the indices, are invariant, *i.e.* belong to the representation $(0, 0)$. The four dimensional gamma matrices are (in the chiral representation)

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{2.11}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.12}$$

The Dirac spinors are given by

$$\psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}: \tag{2.13}$$

they are equivalent to two independent Weyl (two-component) spinors; while the Majorana spinors have the form

$$\psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.14}$$

The Dirac conjugates of these are

$$\bar{\psi}_D \equiv \psi_D^\dagger \gamma^0 = (-\psi^\alpha, -\bar{\chi}_{\dot{\alpha}}); \tag{2.15}$$

the Majorana spinors satisfies

$$\bar{\psi}_M = (-\chi^\alpha, -\bar{\chi}_{\dot{\alpha}}) = \psi_M^T \mathcal{C}, \tag{2.16}$$

where

$$\mathcal{C} = -i\,\gamma^0\,\gamma^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix} \tag{2.17}$$

is a real matrix. We use the convention

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = -\chi_\alpha\psi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi, \tag{2.18}$$

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha} = \bar\chi\bar\psi. \tag{2.19}$$

Namely, spinors are anticommuting (Grassmannian) quantities. Other useful formulas are

$$(\psi\,\chi)^\dagger = \bar\psi\,\bar\chi \tag{2.20}$$

$$\bar\psi_D\,\psi_D = -\psi\,\chi - \bar\psi\,\bar\chi, \tag{2.21}$$

$$\bar\psi_D\,i\gamma^\mu\,\partial_\mu\psi_D = -i\,\psi\,\sigma^\mu\,\partial_\mu\bar\psi - i\,\bar\chi\,\bar\sigma^\mu\,\partial_\mu\chi, \tag{2.22}$$

$$(\bar\sigma)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\,\epsilon^{\alpha\beta}(\sigma)_{\beta\dot\beta}. \tag{2.23}$$

Looking at the Weyl equation

$$i\,\sigma^\mu\,\partial_\mu\bar\psi = 0 \tag{2.24}$$

$$E\,\bar\psi = i\,\partial_0\bar\psi = -\sigma\cdot\mathbf{p}\,\bar\psi \tag{2.25}$$

one sees that $E > 0$ implies $\sigma\cdot\mathbf{p} < 0$: $\bar\psi$ is the wave function of a spinor of negative helicity. Vice versa, χ describes a positive helicity spinor. As an operator χ creates negative helicity (or destroy a positive-helicity) state.

3. Supersymmetry algebra

Supersymmetry charges are

$$Q_\alpha = \begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix}. \tag{3.1}$$

In the four-spinor notation ($\mathcal{C} = i\gamma^2\gamma^0$)

$$\bar Q \equiv \mathcal{C}Q^T = -\begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix}\begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix} = -\begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix} \tag{3.2}$$

$$\{Q_\alpha, \bar Q_\beta\} = 2\begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta}P_\mu \\ \bar\sigma^{\mu\dot\alpha\alpha}P_\mu & 0 \end{pmatrix} \tag{3.3}$$

From now on however we switch to two-component spinor notation. $\mathcal{N} = 1, 2, \ldots 4$ susy algebra are ($i = 1, 2, \ldots \mathcal{N}$)

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = -2\,\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{ij}, \qquad \{Q^i_\alpha, Q^j_\beta\} = \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 0 \quad (*), \tag{3.4}$$

$$[P^\mu, Q] = [P^\mu, \bar{Q}] = 0, \qquad [P^\mu, P^\nu] = 0, \tag{3.5}$$

$$i[M^{\mu\nu}, Q_\alpha] = (\sigma^{\mu\nu} Q)_\alpha, \qquad (\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{1}{4}[\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu]_\alpha^{\ \beta}, \tag{3.6}$$

$$i[M^{\mu\nu}, \bar{Q}^{\dot\alpha}] = (\bar\sigma^{\mu\nu} \bar{Q})^{\dot\alpha}, \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{1}{4}[\bar\sigma^\mu \sigma^\nu - \bar\sigma^\nu \sigma^\mu]^{\dot\alpha}_{\ \dot\beta}, \tag{3.7}$$

The rest is the standard Poincaré group algebra

$$i[M^{\mu\nu}, P^\lambda] = g^{\mu\lambda} P^\nu - g^{\nu\lambda} P^\mu \tag{3.8}$$

$$i[M^{\mu\nu}, M^{\lambda\kappa},] = g^{\mu\lambda} M^{\nu\kappa} + g^{\nu\kappa} M^{\mu\lambda} - g^{\nu\lambda} M^{\mu\kappa} - g^{\nu\lambda} M^{\mu\kappa} \tag{3.9}$$

where the metric tensor is taken to be

$$\mathrm{diag}\, g^{\mu\nu} = (1, -1, -1, -1) \tag{3.10}$$

(Bjorken-Drell).

3.1. Note

(i) [Haag-Sohnius-Lopszanski theorem]: supersymmetry algebra is the only algebra which contains the Poincaré algebra, which generalizes it by additional Grassmann algebra, and that is consistent with nontrivial S-matrix of quantum field theory. This evades and in a sense generalizes the Coleman-Mandula theorem. The latter states that the only symmetry algebra (without Grassmannian extension) which generalizes the Poincaré algebra to include internal symmetry algebra, and that is consistent with nontrivial S-matrix of quantum field theory, is of the form

$$g \sim g_{Poincare} \otimes g_{internal}, \tag{3.11}$$

a direct product.

(ii) Jacobi's identity reads

$$\{A, \{B, C]] \pm \{B, \{C, A]] \pm \{C, \{A, B]] = 0. \tag{3.12}$$

where either the anticommutator should be chosen when both operators are fermionic, and the signs are determined by whether in that term the Grassmannian operators appear in the even (+) or odd (−) permution with respect to the first term. For instance, if $A =$ bosonic, B, C are fermionic, then it reads

$$[A, \{B, C\}] + \{B, [C, A]\} - \{C, [A, B]\} = 0. \tag{3.13}$$

(iii) For $\mathcal{N} = 2$ it allows the supersymmetry algebra to be generalized with central extension,

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta}\, \epsilon^{ij}\, (U + iV) \tag{3.14}$$

where U, V are the central charges ($i.e$, they commute with all the generators.) It has been shown explicitly (Olive, Witten) that in $\mathcal{N} = 2$ non-Abelian gauge theories U, V do appear and correspond to the electric and magnetic charges.

3.2. One particle representations

Let us consider now the representation of supersymmetry algebra on one particle states.

(i) For a massive $\mathcal{N} = 1$ supersymmetric particle states, one has ($P^\mu = (M, 0, 0, 0)$)

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = \delta_{\alpha\dot\alpha}\, 2M, \qquad \alpha, \dot\alpha = 1, 2, \tag{3.15}$$

or, by defining

$$b^\dagger_\alpha = \frac{1}{\sqrt{2M}} Q_\alpha, \qquad b_{\dot\alpha} = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot\alpha}. \tag{3.16}$$

These can be regarded as two pairs of annihilation and creation operators, $\{b_{\dot\alpha}, b^\dagger_\alpha\} = \delta_{\alpha\dot\alpha}$. The complete set of one particle states can then be conctructed by defining the vacuum state by ($i = 1, 2$)

$$b_i\, |0\rangle = 0; \tag{3.17}$$

the full set of states are

$$|0\rangle, \quad b^\dagger_1|0\rangle, \quad b^\dagger_2|0\rangle, \quad b^\dagger_1 b^\dagger_2|0\rangle, \tag{3.18}$$

they form a degenerate supersymmetry multiplet (two bosons and two fermions).

For $\mathcal{N}$ supersymmetry, the same argument shows that the multiplicity of a massive multiplet is

$$\sum_{n=0}^{2\mathcal{N}} \binom{2\mathcal{N}}{n} = 2^{2\mathcal{N}}. \tag{3.19}$$

(ii) Massless $\mathcal{N} = 1$ supersymmetric particle states: In this case it is not possible to go to the rest frame but the momentum can be chosen as $P^\mu = (p, 0, 0, p)$. Then

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = \begin{pmatrix} 2p & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\alpha}} \tag{3.20}$$

The state $b_2^\dagger|0\rangle$ have a zero norm. The particle states are given by the positive norm states, half of (3.18),

$$|0\rangle, \quad b_1^\dagger|0\rangle. \tag{3.21}$$

The multiplicily of a massless $\mathcal{N} = 1$ supersymmetry multiplet is

$$\sum_{n=0}^{\mathcal{N}} \binom{2\mathcal{N}}{n} = 2^{\mathcal{N}}. \tag{3.22}$$

(iii) Massive $\mathcal{N} = 2$ supersymmetric particle states with central charges. In the rest frame ($P^\mu = (M, 0, 0, 0)$) the supersymmetry algebra reduces to

$$\{Q^i_\alpha, \bar{Q}^j_{\dot{\alpha}}\} = \delta^{ij}\, \delta_{\alpha\dot{\alpha}}\, 2M, \qquad \alpha, \dot{\alpha} = 1, 2, \quad i, j = 1, 2, \tag{3.23}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta}\, \epsilon^{ij}\, (U + iV) \tag{3.24}$$

Within an irreducible representation U and V are just numbers (electric and magnetic charges of these particles). There are three cases:

$2M < \sqrt{U^2 + V^2}$: It is not possible to find a positive-norm representation of the algebra;

$2M = \sqrt{U^2 + V^2}$: A representation exists with multiplicity $2^{\mathcal{N}} = 4$ (*short multiplet*) ("BPS" saturated case);

$2M > \sqrt{U^2 + V^2}$: A representation exists with multiplicity $2^{2\mathcal{N}} = 16$ (*long multiplet*).

Proof: Define

$$\frac{Q^1_1}{\sqrt{2M}} = b_1 \qquad \frac{Q^1_2}{\sqrt{2M}} = b_2 \qquad \frac{Q^2_1}{\sqrt{2M}} = b_3 \qquad \frac{Q^2_2}{2M} = b_4 \tag{3.25}$$

$$-\frac{U}{\sqrt{2M}} = u \qquad -\frac{V}{\sqrt{2M}} = v \tag{3.26}$$

then

$$\{b_i, b_j^\dagger\} = \delta_{ij} \qquad \{b_1, b_4\} = u + iv \qquad \{b_2, b_3\} = -u - iv \tag{3.27}$$

$$\{b_1^\dagger, b_4^\dagger\} = u - iv \qquad \{b_2^\dagger, b_3^\dagger\} = -u + iv \tag{3.28}$$

Now make the change of variables

$$Q_\alpha^1 \longrightarrow e^{i\gamma}\, Q_\alpha^1 \qquad Q_\alpha^2 \longrightarrow Q_\alpha^2 \tag{3.29}$$

$$b_1 \longrightarrow e^{i\gamma}\, b_1 \qquad b_2 \longrightarrow e^{i\gamma}\, b_2 \tag{3.30}$$

to have $\{b_1, b_4\}$ real and positive:

$$\{b_1, b_4\} = \{b_1^\dagger, b_4^\dagger\} = \alpha = \frac{\sqrt{U^2 + V^2}}{2M} \tag{3.31}$$

$$\{b_2, b_3\} = \{b_2^\dagger, b_3^\dagger\} = -\alpha \tag{3.32}$$

In order to see the spectrum, it is convenient to set

$$A = b_1 \cos\vartheta + b_4^\dagger \sin\vartheta \qquad B = -b_1 \sin\vartheta + b_4^\dagger \cos\vartheta. \tag{3.33}$$

The condition $\{A, B\} = \{A, B^\dagger\} = 0$ yields $\vartheta = \frac{\pi}{4}$: A and B satisfy separate anticommutators

$$\{A, B\} = 0 \qquad \{A, A^\dagger\} = 1 + \alpha \qquad \{B, B^\dagger\} = 1 - \alpha \tag{3.34}$$

Thus if $|\alpha| < 1$ there are two creation operators $A^\dagger$, $B^\dagger$, while if $\alpha = \pm 1$ $B^\dagger$ (or $A^\dagger$) creates zero-norm states and only $A^\dagger$ (or $B^\dagger$) survives. Repeating the same passages for b_2 and $b_3^\dagger$ leads to an identical result.

The net result is that the absence of negative norm states requires $|\alpha| \leq 1$, or

$$M \geq \frac{\sqrt{U^2 + V^2}}{2}. \tag{3.35}$$

Particles with mass $M > \frac{\sqrt{U^2+V^2}}{2}$ come in "long multiplets" with multiplicity $2^{2\mathcal{N}} = 8$, while the BPS particles with mass $M = \frac{\sqrt{U^2+V^2}}{2}$ come in "short multiplets" of multiplicity $2^{\mathcal{N}} = 4$.

3.3. Field transformations

Now we wish to find the representation of supersymmetry algebra in the space of fields rather than in one-particle states. Now the operators appearing in the algebra

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = -2\,\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{ij}, \qquad P_\mu = i\partial_\mu \tag{3.36}$$

act on the space of fields. Introduce superfields living in superspace (Salam and Stradthee 1974), considering them as power series in the coordinates $\theta, \bar\theta$:

$$F(x,\theta,\bar\theta) = f(x) + \theta\,\psi(x) + \bar\theta\,\bar\chi(x) + \theta\theta\, m(x) + \bar\theta\bar\theta\, n(x) + \\ + \theta\sigma^\mu\bar\theta\, v_\mu(x) + \theta\theta\,\bar\theta\,\bar\lambda(x) + \bar\theta\bar\theta\,\theta\,\phi(x) + \theta\theta\,\bar\theta\bar\theta D(x) \tag{3.37}$$

where θ^α and $\bar\theta_{\dot\beta}$ are Grassmannian coordinates,

$$\{\theta_1, \theta_2\} = 0, \quad (\theta_1)^2 = (\theta_2)^2 = 0, \quad \theta\theta \equiv \theta^\alpha\theta_\alpha = 2\theta^2\theta^1, \quad \bar\theta\bar\theta \equiv \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = 2\bar\theta_1\bar\theta_2 \tag{3.38}$$

etc. For Grassmannian coordinates the integration and derivation is the same:

$$\int d\theta = \frac{\partial}{\partial\theta}, \qquad \int d\theta^1\theta^1 = \int d\theta^2\theta^2 = 1, \qquad \int d\theta^1\, 1 = 0, \qquad \delta(\theta) = \theta, \tag{3.39}$$

$$\int d^2\theta\,\theta\theta = 1, \qquad \int d^2\bar\theta\,\bar\theta\bar\theta = 1, \qquad \int d^2\theta\, 1 = \int d^2\theta\,\theta^1 = 0, \tag{3.40}$$

where $d^2\theta$ is defined as $\frac{1}{2}d\theta^1 d\theta^2$. The same rules apply to $\bar\theta$, with $d^2\bar\theta = \frac{1}{2}d\bar\theta_2 d\bar\theta_1$. Note that the differentials of Grassmannian variables are Grassmannian and anticommute. In the expansion of $F(x,\theta,\bar\theta)$ in powers of $\theta, \bar\theta$ there is a finite number of terms because $\theta^3 = \bar\theta^3 = 0$. The fields that appear in the expansion are called component fields. It is possible to see supersymmetry transformations on these fields as translations in superspace:

$$G(x,\theta,\bar\theta) = e^{i(-x^\mu P_\mu + \theta Q + \bar\theta\bar Q})\tag{3.41}$$

and using Baker-Hausdorff formula

$$G(0,\xi,\bar\xi)G(x,\theta,\bar\theta) = G(x^\mu + i\theta\sigma^\mu\bar\xi - i\xi\sigma^\mu\bar\theta, \theta+\xi, \bar\theta+\bar\xi) \tag{3.42}$$

Then the infinitesimal translation operator in superspace is $\xi Q + \bar\xi\bar Q$, where

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \tag{3.43}$$

$$Q^{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\partial_{\mu} \tag{3.44}$$

and its action on superfields is

$$\delta_{\xi,\bar{\xi}}F = [\xi Q + \bar{\xi}\bar{Q}, F] = \delta f(x) + \theta\,\delta\psi(x) + \bar{\theta}\,\delta\bar{\chi}(x) + \theta\theta\,\delta m(x) + \ldots \tag{3.45}$$

The above equation defines the action of infinitesimal supersymmetry transformations on component fields. Note that these representations are in general reducible; the usual way to obtain irreducible ones is to find covariant constraints which reduces the number of component fields. Note that these constraints must not impose conditions on the x-dependence of the component fields. The most important examples of constraints of this kind are chiral and vector superfields. Before introducing them, we define a covariant derivative in superspace:

$$D_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} + i(\sigma^{\mu}\theta)_{\alpha}\partial_{\mu} \tag{3.46}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} \tag{3.47}$$

which satisfies

$$\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} \qquad \{Q, D\} = \{Q, \bar{D}\} = \{\bar{Q}, D\} = \{\bar{Q}, \bar{D}\} = 0 \tag{3.48}$$

$$(D)^3 = (\bar{D})^3 = 0 \tag{3.49}$$

3.4. Chiral superfields

A chiral superfields is defined by

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \qquad \text{(chiral)} \tag{3.50}$$

$$D_{\alpha}\Phi = 0 \qquad \text{(antichiral)} \tag{3.51}$$

This constraint is easier to understand if we introduce chiral coordinates

$$y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta} \qquad y^{\dagger\mu} = x^{\mu} - i\theta\sigma^{\mu}\bar{\theta} \tag{3.52}$$

and change coordinates $(x, \theta, \bar{\theta}) \longrightarrow (y, \theta, \bar{\theta})$; in the new coordinates

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \equiv -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}|_{y}, \tag{3.53}$$

and the constraint (3.50) can be solved by the functions of the form

$$\Phi = \Phi(y, \theta) = A(y) + \sqrt{2}\theta\,\psi(y) + \theta\theta\,F(y) \tag{3.54}$$

or, going back to the variables $(x, \theta, \bar{\theta})$,

$$\Phi = A(x) + \sqrt{2}\theta\,\psi(x) + \theta\theta\,F(x) + i\theta\sigma^\mu\bar{\theta}\,\partial_\mu A(x) - \frac{i}{\sqrt{2}}\theta\theta\,\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \frac{1}{4}\theta\theta\,\bar{\theta}\bar{\theta}\,\Box A(x). \tag{3.55}$$

$\theta, \bar{\theta}$ have the dimension of $-\frac{1}{2}$, so if $A(x)$ is a scalar field of dimension 1 $\psi(x)$ is a spinor field of dimension $\frac{3}{2}$; $F(x)$ is a scalar field of a wrong dimension (2). In fact, we shall see that it is an auxiliary field, entering the Lagrangean without derivatives, and consequently can be eliminated through the equations of motion.

Chiral superfields for a ring under addition and product,

$$\bar{D}_{\dot{\alpha}}(\Phi_1\Phi_2) = \bar{D}_{\dot{\alpha}}(\Phi_1)\Phi_2 + \Phi_1\bar{D}_{\dot{\alpha}}\Phi_2 = 0 \qquad \bar{D}_{\dot{\alpha}}(\Phi_1 + \Phi_2) = \bar{D}_{\dot{\alpha}}\Phi_1 + \bar{D}_{\dot{\alpha}}\Phi_2 = 0 \tag{3.56}$$

The product $\Phi^\dagger\Phi$ is not a chiral superfield, because $\bar{D}_{\dot{\alpha}}\Phi^\dagger \neq 0$. $\Phi^\dagger\Phi$ is a real superfield: it can be used in constructing supersymmetric Lagrangians, as we see later.

Upon applying a supersymmetry transformation to a superfield, a component field of dimension n gets transformed to $\delta_{\xi,\bar{\xi}}\zeta(x) = \xi\,\delta\zeta(x) + \bar{\xi}\,\bar{\delta}\zeta(x)$ containing terms that have dimension $n+\frac{1}{2}$, as ξ and $\bar{\xi}$ carry the dimension $-\frac{1}{2}$. These terms can be fields of higher dimension, or derivatives of fields of lower dimension. It is easy to see that the $\theta\theta$ term (usually called *F-term*) of a chiral superfield or a composite theirof, always transforms into is the spacetime derivative. The same can be shown for the highest $\theta\theta\bar{\theta}\bar{\theta}$ term (also called *D-term*) of a generic superfield. These observations are used to construct a supersymmetric action with a given set of superfields.

3.5. Vector superfields

Another possible covariant constraint on a superfield V is

$$V^\dagger = V \tag{3.57}$$

A superfield of this kind is called a vector (or real) superfield. The generic form for V is

$$V = C(x) + i\theta\,\chi(x) - i\bar{\theta}\,\bar{\chi}(x) + \frac{i}{2}\theta\theta\,[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}\,[M(x) - iN(x)] + \\ -\theta\sigma^{\mu}\bar{\theta}\,v_{\mu}(x) + i\theta\theta\,\bar{\theta}\,[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)] - i\bar{\theta}\bar{\theta}\,\theta\,[\lambda(x) - \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)] + \\ +\frac{1}{2}\theta\theta\,\bar{\theta}\bar{\theta}\,[D(x) + \frac{1}{2}\Box C(x)] \quad (3.58)$$

with all the bosonic fields real. Chiral superfields can be used to describe describe quarks and leptons (and their superpartnes), while vector superfields can be used to describe the gauge bosons.

As $\Phi + \Phi^{\dagger}$ is a real superfield. the concept of gauge transformation can be generalized to supersymmetric gauge transformations,

$$V \longrightarrow V + \Phi + \Phi^{\dagger} \quad (3.59)$$

which for the $\theta\bar{\theta}$ term reads

$$v_{\mu} \longrightarrow v_{\mu} - i\partial_{\mu}(A - A^{\dagger}) \quad (3.60)$$

which is the ordinary Abelian gauge transformation. In nonabelian gauge theories the supersymmetric extension of gauge transformation takes a more complicated form.

There is a gauge (the Wess-Zumino gauge) in which most of the components of the vector superfield are zero:

$$V = -\theta\sigma^{\mu}\bar{\theta}\,v_{\mu}(x) + i\theta\theta\,\bar{\theta}\,\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\,\theta\,\lambda(x) + \frac{1}{2}\theta\theta\,\bar{\theta}\bar{\theta}\,D(x) \quad (3.61)$$

In this gauge $V^3 = 0$ and supersymmetry is broken, but it is still possible a usual gauge transformation like $v_{\mu} \longrightarrow v_{\mu} + \partial_{\mu}\Lambda$. Now we want to find an equivalent of the field strength. The lower-dimensional gauge invariant component field is $\lambda(x)$, so we can try

$$W_{\alpha} = -\frac{1}{4}\bar{D}\bar{D}D_{\alpha}V \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V \quad (3.62)$$

which are gauge-invariant chiral superfields; in fact $\bar{D}_{\dot{\alpha}}W_{\beta} = D_{\alpha}\bar{W}_{\dot{\beta}} = 0$ because $D^3 = \bar{D}^3 = 0$. They also satisfy $D^{\alpha}W_{\alpha} = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$. If we switch to chiral coordinates W_α assume the form

$$W_\alpha(y,\theta) = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha(\partial_\mu v_\nu(y) - \partial_\nu v_\mu(y)) + \theta\theta(\sigma^\mu\partial_\mu\bar\lambda(y))_\alpha \tag{3.63}$$

so it contains the field strength $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$, λ and D which are gauge-invariant fields, and it is a good candidate for a superfield strength.

4. Supersymmetric gauge theories

The simplest nontrivial supersymmetric theory is the Wess-Zumino model for chiral fields, in which the Lagrangian kinetic term is constructed from

$$\Phi^\dagger\Phi|_{\theta\theta\bar\theta\bar\theta} = F^\dagger F + \frac{1}{4}A^\dagger\Box A + \frac{1}{4}\Box A^\dagger A - \frac{1}{2}\partial_\mu A^\dagger\partial^\mu A + \frac{i}{2}\partial_\mu\bar\psi\bar\sigma^\mu\psi - \frac{i}{2}\bar\psi\bar\sigma^\mu\partial_\mu\psi \tag{4.1}$$

which contains the usual kinetic terms for scalar and spinor fields. The complete Lagrangian is

$$\mathcal{L} = \sum_i \Phi_i^\dagger\Phi_i|_D + \mathcal{P}(\Phi)|_F + h.c. \tag{4.2}$$

where $\mathcal{P}$ is the superpotential. If we restrict to renormalizable interactions [1] we have

$$\mathcal{P} = \frac{m_{ij}}{2}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\,\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i \tag{4.4}$$

In terms of component fields, it reads (for a single chiral field)

$$\mathcal{L} = -i\bar\psi\bar\sigma\partial\psi + \partial_\mu A^\dagger\partial^\mu A - (\frac{m}{2}\psi\psi + h.c.) - (g\psi\psi A + h.c.) - V(A, A^\dagger) \tag{4.5}$$

$$V = F^\dagger F = |\lambda + mA + gA^2|^2. \tag{4.6}$$

For a vector superfield a kinetic Lagrangian can be constructed from the term

$$W^\alpha W_\alpha|_F = -2i\lambda\sigma^\mu\partial_\mu\bar\lambda - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}F^{\mu\nu}\tilde F_{\mu\nu} + D^2 \tag{4.7}$$

Now it is easy to construct the Lagrangian of the supersymmetric quantum electrodynamics (SQED). We have left and right-handed electrons and positrons,

[1] If the condition of renormalizability is not needed (*e.g.* effective action) the supersymmetric Lagrangian of chiral superfields have the general form,

$$\int d^4x\, d^2\theta\, d^2\bar\theta\, K(\Phi^\dagger,\Phi) = \int d^4x \frac{\partial^2 K}{\partial\phi_i^\dagger\partial\phi_j}\,\partial_\mu\phi_i^\dagger\partial^\mu\phi_j + \dots \tag{4.3}$$

i.e. it is given in terms of a Kähler potential $K(\Phi^\dagger,\Phi)$.

so we must have two chiral fields in the theory, Φ and $\tilde{\Phi}$. We introduce a global $U(1)$ transformation corresponding to the electric charge

$$\Phi \longrightarrow e^{-ie\Lambda}\Phi \qquad \tilde{\Phi} \longrightarrow e^{ie\Lambda}\tilde{\Phi} \tag{4.8}$$

and consider the Lagrangian invariant under this transformation, that will generally contain only the mass term in the superpotential. When we see Λ as a function of spacetime coordinates, we must promote it to a full chiral multiplet to transform a chiral field into a chiral field; but then the kinetic term transforms as

$$\Phi^\dagger\Phi \longrightarrow \Phi^\dagger e^{-ie(\Lambda-\Lambda^\dagger)}\Phi \tag{4.9}$$

We must introduce a vector gauge field with the transformation property

$$V \longrightarrow V + i(\Lambda - \Lambda^\dagger) \tag{4.10}$$

so as to make a gauge invariant such as $\Phi^\dagger e^{eV}\Phi$. Now the complete SQED Lagrangian is

$$\mathcal{L}^{SQED} = \frac{1}{4}(W^\alpha W_\alpha|_F + h.c.) + (\Phi^\dagger e^{eV}\Phi + \tilde{\Phi}^\dagger e^{-eV}\tilde{\Phi})|_D + m(\Phi\tilde{\Phi}|_F + h.c.) \tag{4.11}$$

Actually, another term (Fayet-Iliopoulos term)

$$V|_D \tag{4.12}$$

can be added without breaking the gauge symmetry. Also, a superpotential $\mathcal{P}(\Phi)|_F + h.c.$, can be added.

These results can be generalized to non-abelian gauge theories such as SQCD. We choose a set $\{T^a\}_{a=1\dots N_c^2-1}$ of $N_c \times N_c$ hermitian matrices that belong to the fundamental representation of the Lie algebra of $SU(N_c)$

$$\mathrm{Tr}\, T^a = 0, \qquad \mathrm{Tr}\,(T^aT^b) = \frac{1}{2}\delta_{ab} \qquad [T^a, T^b] = \sum_c if^{abc}T^c \tag{4.13}$$

V is a matrix $V = \sum_a T^aV^a$ and the field strength is defined as

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V. \tag{4.14}$$

The gauge transformation (4.10) in this case takes the form,

$$e^V \longrightarrow e^{i\Lambda^{a\dagger}T^a}\, e^V\, e^{-i\Lambda^a T^a}; \tag{4.15}$$

Figure 1:

while the chiral superfields Q_i, $\tilde{Q}_i$ transform as

$$Q_i \longrightarrow e^{i\Lambda^a T^a} Q_i, \quad Q_i^\dagger \longrightarrow Q_i^\dagger e^{-i\Lambda^{a\dagger} T^a}, \quad \tilde{Q}_i \longrightarrow e^{-i\Lambda^a T^a} \tilde{Q}_i, \quad \tilde{Q}_i^\dagger \longrightarrow \tilde{Q}_i^\dagger e^{i\Lambda^{a\dagger} T^a}. \tag{4.16}$$

A gauge invariant SQCD Lagrangian takes the form,

$$\mathcal{L}^{SQCD} = \frac{1}{16g^2}(W^{a\alpha}W^a_\alpha|_F + h.c.) + (Q_i^\dagger e^V Q_i + \tilde{Q}_i^\dagger e^{-V} \tilde{Q}_i)|_D + (\mathcal{P}(Q,\tilde{Q})|_F + h.c.). \tag{4.17}$$

This Lagrangian takes its usual form after eliminating the auxiliary fields and after a rescaling $V \longrightarrow 2\,g\,V$.

5. Nonrenormalization theorem and NSVZ β function

5.1. Superspace propagator and nonrenormalization theorem

The Wess-Zumino lagrangian is

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left(\Phi^\dagger \Phi + \frac{1}{2} m\, \Phi^2 \delta^2(\bar{\theta}) + h.c. + g\, \Phi^3 \delta^2(\bar{\theta}) + h.c. \right) \tag{5.1}$$

The superfield propagator is

$$\langle T\{\Phi(x,\theta,\bar{\theta})\Phi(x',\theta',\bar{\theta}')\}\rangle = -m\delta^2(\theta-\theta')e^{-i(\theta\sigma^\mu\bar{\theta}-\theta'\sigma^\mu\bar{\theta}')\partial_\mu}\Delta_c(x-x') \tag{5.2}$$

$$\langle T\{\Phi(x,\theta,\bar{\theta})\Phi^\dagger(x',\theta',\bar{\theta}')\}\rangle = e^{-i(\theta\sigma^\mu\bar{\theta}+\theta'\sigma^\mu\bar{\theta}'-2\theta\sigma^\mu\theta')\partial_\mu}\Delta_c(x-x') \tag{5.3}$$

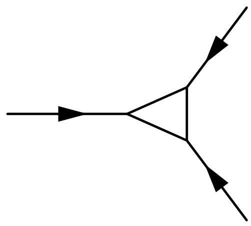
where $\Delta_c(x-x')$ is the usual 2-point function. Consider loop corrections

$$\propto \quad \delta^2(\theta-\theta')\delta^2(\theta-\theta') = \delta^2(\theta-\theta')\delta^2(0) = 0 \tag{5.4}$$

while that of Fig. (2) is nonvanishing and gives the wave function renormaliza-

Figure 2:

tion. Similarly the chiral graph of



$$\propto \delta^2(0) = 0 \tag{5.5}$$

Note that no superpotential term can be generated by perturbative corrections. Only D-terms can be generated.
In general, if Φ is a chiral superfield,

$$\langle \Phi \cdots \Phi \rangle = 0 \qquad \text{perturbatively} \tag{5.6}$$

that is it receives zero contribution from any order of perturbation theory; the only contributions come from those terms which are present in the original (tree-level) lagrangian, such as $g\Phi^3$ in the Wess-Zumino model. In other words there are no terms generated by supergraphs with loops. Thus, after wave-function renormalization, we have (the subscript R stays for "renormalized")

$$g\Phi^3 = g_R \Phi_R^3 \tag{5.7}$$

$$\Phi = Z^{-\frac{1}{2}} \Phi_R \Rightarrow g = Z^{\frac{3}{2}} g_R \tag{5.8}$$

5.2. Nonrenormalization theorem and anomalous and non-anomalous symmetries

The above proof is essentially perturbative (diagrammatic). Certain superpotentials are protected by some symmetry, such as $U(1)$, $SU(n_f)$, etc. If such symmetry is exact and non-anomalous, then these superpotential are not renormalized both perturbatively and non-perturbatively. If such terms are absent in the Lagrangian, they cannot be generated by quantum effects.

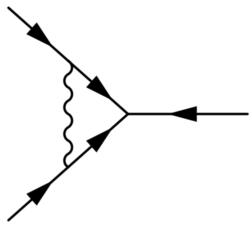
If instead this symmetry is anomalous (as in $U_A(1)$ in QCD-like theories), then terms violating non-renormalization theorems can be generated nonperturbatively, *e.g.* by instantons. This will be the subject of the next Section.

5.3. Perturbative anomalies due to massless loops

If however the theory contains massless particles, the non-renormalization theorem can be violated apparently. Consider a $\mathcal{N} = 1$ Yang=Mills lagrangian [?]

$$\mathcal{L} = \bar{\Phi}^a (e^V)_{ab} \Phi^b \Big|_D + \frac{1}{g^2} W^\alpha W_\alpha \Big|_F + h.c. + d_{abc} \Phi^a \Phi^b \Phi^c \Big|_F + h.c. \tag{5.9}$$

Consider the graph



$$\sim \int d^2\theta d^2\bar{\theta} \frac{\Phi^2 D^2 \Phi}{\Box} \tag{5.10}$$

This is a non local term. In a background field framework, when external momenta tends to 0, $D^2 = \left(\frac{\partial}{\partial\theta} + i\bar{\theta}\sigma^\mu\partial_\mu\right)^2 \sim \bar{\theta}^2 \Box$ and

$$\int d^2\theta d^2\bar{\theta} \frac{\Phi^2 D^2 \Phi}{\Box} \sim \int d^2\theta \Phi^3 \tag{5.11}$$

which imitates a F-term. Clearly this process is not possible if we ca write a Wilsonian action S_W and we can make all quantities infrared-free.
It is important to note that, in general, we have a non-analytic function of original couplings in front of 5.11. Let's take, for example, the Wess-Zumino model

$$\mathcal{L} = \int d^8 z \sum_i \bar{\Phi}_i \Phi_i + \int d^6 z P + h.c. \tag{5.12}$$

$$P = \lambda_1 \Phi_1^3 + \lambda_2 \Phi_1 \Phi_2^2 + \mu \Phi_1 \tag{5.13}$$

It can be shown that the superpotential receives a 2-loop contribution

$$\sim (\lambda_1^\star)^2 \lambda_2^3 \Phi_2^3. \tag{5.14}$$

5.4. Gauge kinetic terms and generalized non-renormalization theorem

Gauge kinetic terms in gauge theories provides another kind of subtlety in the consideration of non-renormalization theorem.

$$\begin{aligned}\int d^4x \int d^2\theta W^\alpha W_\alpha &= \frac{1}{16}\int d^4x \int d^2\theta (\bar{D}^2 e^{-V} D^\alpha e^V)(\bar{D}^2 e^{-V} D_\alpha e^V) \\ &= \frac{1}{16}\int d^4x \int d^2\theta \bar{D}^2 (e^{-V} D^\alpha e^V)(\bar{D}^2 e^{-V} D_\alpha e^V) \\ &= \frac{1}{4}\int d^4x \int d^2\theta \int d^2\bar{\theta} (e^{-V} D^\alpha e^V)(\bar{D}^2 e^{-V} D_\alpha e^V) \qquad (5.15)\end{aligned}$$

and it is really a D-term, so it may receive contributions from radiative corrections. However, it turns out that graphs higher than 1 loop do not contribute. (More precisely, it has been shown (see [?]) in the framework of Wilsonian action, where all quantities are infrared-free, while in a standard background field method there are in general multiloop corrections). This generalized non-renormalization theorem played important role in the subsequence developments.

5.5. NSVZ β function in $N = 1$ supersymmetric gauge theories

The bare Lagrangian of an $N = 1$ supersymmetric gauge theory with generic matter content is given by

$$L = \frac{1}{4}\int d^2\theta \left(\frac{1}{g_h^2(M)}\right) W^a W^a + h.c. + \int d^4\theta \sum_i \Phi_i^\dagger e^{2V_i} \Phi_i \qquad (5.16)$$

where

$$\frac{1}{g_h^2(M)} = \frac{1}{g^2(M)} + i\frac{\theta(M)}{8\pi^2} \equiv i\frac{\tau(M)}{4\pi} \qquad (5.17)$$

and $g(M)$ and $\theta(M)$ stand for the bare coupling constant and vacuum parameter, M being the ultraviolet cutoff. "h" stays for "holomorphic" (we will omit it in the sequel). Note that with this convention the vector fields $A_\mu(x)$ and the gaugino (gluino) fields $\lambda_\alpha(x)$ contain the coupling constant and hence, in accordance with the non Abelian gauge symmetry, are not renormalized. By a generalized nonrenormalization theorem [?] the effective Lagrangian at scale μ takes the form,

$$L = \frac{1}{4}\int d^2\theta \left(\frac{1}{g^2(M)} + \frac{b_0}{8\pi^2}\log\frac{M}{\mu}\right) W^a W^a + h.c. + \int d^4\theta \sum_i Z_i(\mu, M)\Phi_i^\dagger e^{2V_i}\Phi_i \,, \tag{5.18}$$

(plus higher dimensional terms). Here

$$b_0 = -3N_c + \sum_i T_{Fi}; \qquad T_{Fi} = \frac{1}{2} \qquad \text{(quarks)}\,. \tag{5.19}$$

Novikov et. al. then invoked the 1PI effective action to define a "physical" coupling constant for which they obtained the well-known β function (Eq.(5.23) below) [?]. Recently the derivation of the NVSZ beta function was somewhat streamlined by Arkani-Hamed and Murayama [?]. (See also [?].) They obtained the NVSZ beta function in the standard Wilsonian framework, without appealing to the 1PI effective action (hence no subtleties due to zero momentum external lines, such as those leading to apparent violation of nonrenormalization theorem[?, ?]). They insist simply that at each infrared cutoff μ the matter kinetic terms be re-normalized so that it resumes the standard canonical form, which is the standard procedure in the Wilsonian renormalization group. But the field rescaling

$$\Phi_i = Z_i^{-1/2}\Phi_i^{(R)} \,, \tag{5.20}$$

introduces necessarily anomalous functional Jacobian [?], and one gets

$$L = \frac{1}{4}\int d^2\theta \left(\frac{1}{g^2(M)} + \frac{b_0}{8\pi^2}\log\frac{M}{\mu} - \sum_i \frac{T_F}{8\pi^2}\log Z_i(\mu, M)\right) W^a W^a + h.c \tag{5.21}$$
$$+ \int d^4\theta \sum_i \Phi_i^{(R)\dagger} e^{2V_i}\Phi_i^{(R)} \equiv \frac{1}{4g^2(\mu)}\int d^2\theta W^a W^a + h.c. + \int d^4\theta \sum_i \Phi_i^{(R)\dagger} e^{2V_i}\Phi_i^{(R)} \,.$$

where

$$\frac{1}{g^2(\mu)} \equiv \frac{1}{g^2(M)} + \frac{b_0}{8\pi^2}\log\frac{M}{\mu} - \sum_i \frac{T_{Fi}}{8\pi^2}\log Z_i(\mu, M)\,. \tag{5.22}$$

This leads to the beta function (call it β_h to distinguish it from the more commonly used definition):

$$\beta_h(g) \equiv \mu\frac{d}{d\mu}g = -\frac{g^3}{16\pi^2}\left(3N_c - \sum_i T_{Fi}(1-\gamma_i)\right) \,, \tag{5.23}$$

where

$$\gamma_i(g(\mu)) = -\mu\frac{\partial}{\partial\mu}\log Z_i(\mu, M)|_{M,g(M)}, \tag{5.24}$$

is the anomalous dimension of the $i-$th matter field. The same result follows by differentiating (5.22) with respect to M with μ and $g(\mu)$ fixed. For SQCD these read

$$\beta_h(g) = -\frac{g^3}{16\pi^2}\left(3N_c - N_f(1-\gamma)\right), \qquad \gamma(g) = -\frac{g^2}{8\pi^2}\frac{N_c^2-1}{N_c} + O(g^4), \tag{5.25}$$

Eq.(5.23) and Eq.(5.25) are the NSVZ β functions [?]. Note that the "holomorphic" coupling constant $g(\mu)$ is a perfectly good definition of the effective coupling constant: it is finite as $M \to \infty$; $\mu =$ finite, and physics below μ can be computed in terms of it. Vice versa, the coupling constant defined as the inverse of the coefficient of W^aW^a in (5.18), $(\frac{1}{g^2(M)} + \frac{b_0}{8\pi^2}\log\frac{M}{\mu})^{-1}$, is *not* a good definition of an effective coupling constant, as long as $N_f \neq 0$: it is divergent in the limit the ultraviolet cutoff is taken to infinity. In other words, the renormalization of the matter fields (5.20) is the standard, compulsory step of renormalization, such that the low energy physics is independent of the ultraviolet cutoff, M. Let us also note that, in spite of its name, the holomorphic coupling constant gets renormalized in a non-holomorphic way, due to the fact that $Z_i(\mu, M)$ is real. Another consequence of the reality of $Z_i(\mu, M)$ is that θ is not renormalized: this is evident from the same RG equation (5.23) written in terms of τ variable ,

$$\mu\frac{d}{d\mu}\tau(\mu) = -\frac{i}{2\pi}\left(3N_c - \sum_i T_{Fi}(1-\gamma_i)\right), \tag{5.26}$$

showing that the NSVZ beta function is essentially perturbative. It is interesting to observe that the above procedure parallels nicely the original derivation by Novikov et. al. of the beta function by use of some instanton-induced correlation functions. More recently the NSVZ β function in $N = 1$ supersymmetric QCD has been rederived by Arnone, Fusi and Yoshida [?], by using the method of exact renormalization group.

5.6. Zero of the NVSZ beta function, Seiberg's duality and CFT in SQCD

For the range of the flavor $\frac{3N_c}{2} < N_f < 3N_c$ (conformal window) Seiberg discovered by using the NVSZ β function that the theory at low-energy is at a nontrivial infrared fixed point [**?**]. At the zero of the β function the anomalous dimension of the matter field is found to be:

$$\gamma(g^*) = \frac{3N_c - N_f}{N_f}. \tag{5.27}$$

It turns out that this result is in agreement with that determined from the superconformal algebra, which contains the non-anomalous $U_R(1)$ symmetry. This and many other consistency checks allowed Seiberg to conclude that in the conformal window, and at the origin of the moduli space (namely, in the theory where all VEV's vanish), the theory flows into a nontrivial infrared fixed point. Such a theory has no particle description, and as such, can be described by more than one type of gauge theory. In fact, in $SU(N_c)$ theory, the theory can be either described as the standard SQCD with N_f flavors, or in terms of a dual theory, which is an $SU(\tilde{N}_c)$ gauge theory with N_f sets of dual quarks, plus singlet meson fileds, where $\tilde{N}_c \equiv N_f - N_c$. They have the same infrared behavior. This is the first example of the $N = 1$ non-Abelian duality, found in many other theories subsequently.

This development enabled Seiberg to complete the picture of dynamical properties of $N = 1$ supersymetric QCD in all cases. Phase, the low-energy effective degrees of freedom, effective gauge group, etc. are summarized in Table 1 (where the bare quark masses are taken to be zero).

6. Anomalies

An important role is played in the analysis of dynamics in gauge theories by various quantum anomalies.

6.1. $U_A(1)$ anomaly

The first example, found by Steinberger, Schwinger, Adler, Bell and Jackiw, is the axial anomaly in QED:

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad J_5^\mu = \bar{\psi} i \gamma_5 \gamma^\mu \psi \tag{6.1}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, and $\epsilon^{0123} = 1$. This appears as the "triagle anomaly": namely due to the linearly divergent triangle fermion 1 loop graph, the three

N_f	Deg.Freed.	Eff. Gauge Group	Phase	Symmetry
0 (SYM)	-	-	Confinement	-
$1 \leq N_f < N_c$	-	-	no vacua	-
N_c	$M, B, \tilde{B}$	-	Confinement	$U(N_f)$
$N_c + 1$	$M, B, \tilde{B}$	-	Confinement	Unbroken
$N_c + 1 < N_f < \frac{3N_c}{2}$	$q, \tilde{q}, M$	$SU(\tilde{N}_c)$	Free-magnetic	Unbroken
$\frac{3N_c}{2} < N_f < 3N_c$	$q, \tilde{q}, M$ or $Q, \tilde{Q}$	$SU(\tilde{N}_c)$ or $SU(N_c)$	SCFT	Unbroken
$N_f = 3N_c$	$Q, \tilde{Q}$	$SU(N_c)$	SCFT (finite)	Unbroken
$N_f > 3N_c$	$Q, \tilde{Q}$	$SU(N_c)$	Free Electric	Unbroken

Table 1: Phases of $N = 1$ supersymmetric $SU(N_c)$ gauge theory with N_f flavors. $M_{ij} = \tilde{Q}_i Q_j$ and $B = \epsilon_{a_1 a_2 \ldots a_{N_c}} \epsilon^{i_1 i_2 \ldots i_{N_c}} Q_{i_1}^{a_1} Q_{i_2}^{a_2} \ldots Q_{i_{N_c}}^{a_{N_c}}$ ($\tilde{B}$ is constructed similarly from the antiquarks, $\tilde{Q}$'s) stand for the meson and baryon like supermultiplets. "Unbroken" means that the full chiral symmetry $G_F = SU_L(N_f) \times SU_R(N_f) \times U(1)$ is realized linearly at low energies. Actually, for $N_f > N_c$ continuous vacuum degeneracy of the theory survives quantum effects, and the entities in the table refers to a representative vacuum at the origin of the quantum moduli space (QMS). For the special case of $N_f = N_c$, the QMS is parametrized by $\det M - B\tilde{B} = \Lambda^{2N_f}$ and the $U(N_f)$ symmetry is the unbroken symmetry of the vaccum with $M = \mathbf{1} \cdot \Lambda^2, \; B = 0$.

point function

$$G^{\mu;\lambda,\nu} \equiv \langle T J_5^\mu(x) J^\lambda(y) J^\nu(z) \rangle \tag{6.2}$$

satisfies an anomalous Ward-Takahashi identity

$$\partial_\mu^x \, G^{\mu;\lambda,\nu} \neq 0, \qquad \text{if} \qquad \partial_\lambda^y \, G^{\mu;\lambda,\nu} = \partial_\nu^z \, G^{\mu;\lambda,\nu} = 0. \tag{6.3}$$

6.2. $\pi_0 \to 2\,\gamma$

Historically, the first physics result involved the process $\pi_{0 \to 2\,\gamma}$, which, in the naïve chiral limit, would bbe zero (Sutherland-Veltman theorem). Let us consider the amplitude,

$$A^{\mu\nu} = \langle \epsilon^\mu(p) \, \epsilon^\nu(k) \, | \partial_\mu \, J_5^\mu(q) \, | 0 \rangle \tag{6.4}$$

where $\epsilon^\mu(p)$, $\epsilon^\nu(k)$ are two external photons, $|0\rangle$ is the vacuum state, and

$$J_5^\mu = \sum_{quarks} i \bar{\psi} \gamma^5 \gamma^\mu \psi \tag{6.5}$$

is the quark axial current. The axial anomaly gives

$$A^{\mu\nu} = \frac{e^2}{16\pi^2} \, C \, \langle \epsilon^\mu(p) \, \epsilon^\nu(k) \, | F\rho\sigma \tilde{F}^{\rho\sigma} \, | 0 \rangle = \frac{e^2}{16\pi^2} \, C \, 2 \, \epsilon_{\mu\nu\rho\sigma} \, p^\rho k^\sigma, \tag{6.6}$$

where

$$C = 2\,N_c\,[\,\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\,] \tag{6.7}$$

C takes into account the electric charges of u and d quarks, their color multiplicity, and pair of graphs in which quarks loop around i the graph in the opposite directions. On the other hand, by assuming that the intermediate state is dominated by a massless pion, one has

$$A^{\mu\nu} = \langle \epsilon^\mu(p)\,\epsilon^\nu(k)\,|\pi\rangle \cdot \frac{1}{q^2}\langle\pi|\partial_\mu J^\mu(q)\,|0\rangle = \langle \epsilon^\mu(p)\,\epsilon^\nu(k)\,|\pi\rangle \cdot \frac{1}{q^2}\cdot q^2 \cdot F_\pi, \tag{6.8}$$

where

$$\langle\pi|\,J^\mu(q)\,|0\rangle = i\,q^\mu F_\pi \tag{6.9}$$

defines the pion decay constant F_π. F_π is known experimentally from the decay process, $\pi \longrightarrow \mu\nu$ to be $F_\pi \simeq 130$ (MeV). Now,

$$A_{\pi\to 2\gamma} = \langle \epsilon^\mu(p)\,\epsilon^\nu(k)\,|\pi\rangle = \frac{N_c\,e^2}{4\pi^2 F_{pi}}\left[\,,\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]\epsilon_{\mu\nu\rho\sigma}\,p^\rho k^\sigma, \tag{6.10}$$

and thus

$$\Gamma(\pi^0 \longrightarrow 2\gamma) = \frac{\alpha^2}{64\pi^3}\frac{m_\pi^3}{F_\pi^2}\,N_c^2\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]^2 \simeq 1\cdot 10^{-5}\,\mathrm{MeV}, \tag{6.11}$$

to be compared with the empirical pion lifetime

$$\tau \simeq (8.4 \pm 0.6)\cdot 10^{-17}\ \mathrm{sec}, \tag{6.12}$$

which implies

$$\Gamma(\pi^0 \longrightarrow 2\gamma) = \frac{\hbar}{\tau} \simeq \frac{1}{1.2}\cdot 10^{-5}\,\mathrm{MeV}. \tag{6.13}$$

Thus the theory predicts the decay width correctly within 20 % which is not bad at all, if one takes into account the extrapolation involved from the massless (soft) pion limit to the on-shell pion mass value.

Note that the factor $N_c^2 \sim 10$ is essential to get such an agreement: this process can be regarded as providing a direct test of the color.

Figure 3:

6.3. $U_A(1)$ anomaly in QCD, Solution of the "$U(1)$ problem".

In QCD, the quark axial current suffers from an axial anomaly due to the gluon fields (as well as due to the photon fields which we neglect in this section):

$$\partial_\mu J_5^\mu = 2N_f \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu\, a}, \qquad J_5^\mu = \bar{\psi} i\gamma_5 \gamma^\mu \psi \tag{6.14}$$

Note that

$$\frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu\, a} = \partial_\mu K^\mu, \qquad K^\mu = \frac{g^2}{16\pi^2}\, \epsilon^{\mu\alpha\beta\gamma}\, \mathrm{Tr}\,(\, F_{\alpha\beta}\, A_\gamma - \frac{2}{3} A_\alpha A_\beta A_\gamma\,) \tag{6.15}$$

so that the axial charge Q_5 might seem to be conserved. Actually in QCD there are nontrivial boundary terms which contribute. In the Euclidean path integral, there are are finite action contributions which behaves asymtotically as

$$A_\mu \sim U^{-1}(x) \partial_\mu U(x), \tag{6.16}$$

where $U(x) \in SU(2) \subset G$, $G = SU(3)$ for QCD. These represent a map $S^3 \to SU(2)$, but since

$$\pi_3(SU(2)) \sim \pi_3(S^3) = \mathbb{Z}, \tag{6.17}$$

they are labeld by an integer winding number (Pontryagin number)

$$\int d^4x \, \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu\, a} = n, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{6.18}$$

The $n = 1$ configuration is called an instanton (see below).

As $\Delta Q_5 \neq 0$ in general, $U_A(1)$ is not conserved, although $SU_L(N_f) \times SU_R(N_f)$ chiral symmetries are respected. In fact, as shown explicitly by 't Hooft, an effective Lagrangian of the form

$$\mathcal{L}_{eff} = \mathrm{const.}\, \epsilon_{i_1, i_2, \ldots i_{N_f}}\, \epsilon^{j_1, j_2, \ldots j_{N_f}}\, \psi_L^{i_1} \cdots \psi_L^{i_{N_f}}\, \bar{\psi}_{R\, j_1} \cdots \bar{\psi}_{R\, j_{N_f}} \tag{6.19}$$

is dynamically (by the instanton) generated. The inti-instanton gives *h.c.* of this. The light Nambu-Golstone boson of the spontaneously broken global $U_A(1)$ is not expected to be there, in contrast to the pions of mass $\sim$ 140 MeV are pseudo Goldstone boson of the broken $SU_A(2)$. This basically solves the $U(1)$ problem. A quantitative explanation of the mass of the "failed Goldstone boson" - the η meson - involves a more careful analysis. In the large N_c approximation it is given by (Witten, Veneziano '79)

$$m_\eta^2 = \frac{4N_f}{F_\pi^2}\,\frac{d^2E}{d^2\theta}|^{YM} = \frac{4N_f}{F_\pi^2}\,\int d^4x\,\langle\,[\,\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu\,a}(x)\,\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu\,a}(0)\,]\rangle^{YM} \tag{6.20}$$

where the correlation function on the right hand side is defined in the pure Yang-Mills theory (i.e., without quarks).

6.4. Nonconservation of chirality and θ term in QCD: the Strong CP Problem

The fact that $\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu\,a}$ is a nontrivial operator means that in the Lagrangian of QCD an extra term

$$\theta\,\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu\,a}. \tag{6.21}$$

Thie term breaks $\mathcal{P}$ and $\mathcal{CP}$. Actually a more carefull definition is needed. As the axial transformation

$$\psi \to e^{i\,\alpha}\psi, \qquad \bar{\psi} \to \bar{\psi}\,e^{i\,\alpha}, \tag{6.22}$$

generates the change of the action

$$\exp iS \to \exp[iS + 2N_f i\,\alpha\int d^4x\,\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu\,a}\,] \tag{6.23}$$

the *tht* parameter is not defined unless the phase convention of the fermions is not fixed. One my choose

$$\mathrm{Arg}\,m_i > 0, \tag{6.24}$$

then the physical parameter is

$$\theta_{ph} = \mathrm{Arg}\,detm + \theta. \tag{6.25}$$

Experimentally,

$$|\theta_{ph}| < 10^{-9}, \tag{6.26}$$

from the absence of the observed electric dipole moment of the neutron.

The problem is then why θ_{ph} is so small ("strong CP problem"). There are several solutions: Peccei-Quinn mechanism and its various variations all of which lead to the existence of a light axion; another possibility is $m_u = 0$.

6.5. Spectral Flow

A way to study the chirality nonconservation is to study the spectral flow. In a gauge background of nonvanishing Pontryagin number n, there are n_- (or n_+) chirality fermion zero modes such that

$$n_- - n_+ = n. \tag{6.27}$$

It is known that in the instanton background ($n = 1$) there is one zero mode of negative chirality

$$\bar{\mathcal{D}}\psi^0 = 0, \qquad \int d^{4x}\, |\psi^0(x)|^2 = 1, \tag{6.28}$$

and no zeromode of positive chirality. In the $A_0 = 0$ gauge, this can be written as

$$(-\frac{\partial}{\partial t} + H^{Dirac})\psi^0 = 0. \tag{6.29}$$

Since ψ^0 is normalizable in (Euclidean) $4D$ it means that at $\tau \to \pm\infty$

$$\psi^0 \to e^{-E(\tau)\tau}, \tag{6.30}$$

with

$$E(\infty) > 0, \qquad E(-\infty) < 0. \tag{6.31}$$

Thus $E(\tau)$ changes the sign of energy as τ varies from $-\infty$ to ∞.

On the other hand, one can show that in the $A_0 = 0$ gauge, the background field $A_\mu(\tau = \infty)$ is a gauge transformation of $A_\mu(\tau = -\infty)$: the spectrum of the Dirac Hamiltonian must be the same at $\tau \to \pm\infty$. It means that the whole energy spectrum gets shifted upwards, and precisely one mode (for each fermion flavor) has moved from the negative Dirac sea to the positive value. For consistency, the three-dimensional Dirac operator $H^{Dirac}(t = 0)$ must possess one ($3D$-normalizable) zeromode, which can be easily found (Kiskis).

6.6. $A_0 = 0$ Gauge

Take

$$U^\dagger = \exp\,[\,i\int_{infty}^{x_4} dx_4 \frac{\sigma_i\,x_i}{x_4^2+\mathbf{x}^2+\rho^2}\,], \tag{6.32}$$

then

$$\frac{i}{g}\,\partial_4\,U^\dagger = -\frac{1}{g}\,\frac{\sigma_i\,x_i}{x_4^2+\mathbf{x}^2+\rho^2}\,U^\dagger = -A_4^{(inst)}\,U^\dagger \tag{6.33}$$

so

$$\tilde{A}_4^{(inst)} = U\,(A_4^{(inst)} + \frac{1}{g}\,\partial_4)\,U^\dagger = 0. \tag{6.34}$$

But then

$$U^\dagger(-\infty,\mathbf{x}) = \mathbf{1}, \qquad U^\dagger(\infty,\mathbf{x}) = e^{i\pi\frac{\sigma_i\,x_i}{\mathbf{x}^2+\rho^2}}. \tag{6.35}$$

$A_i(-\infty,\mathbf{x})^{(inst)} = 0$ and $A_i(\infty,\mathbf{x})^{(inst)} = U(\infty,\mathbf{x})\,\frac{i}{g}\,\partial_i\,U^\dagger(\infty,\mathbf{x})$ are thus obviously related by a gauge transformation in this gauge.

6.7. Chiral anomalies; non-Abelian anomalies; Wess-Zumino consistency condition; Witten's quantization

From the viewpoint that the chiral (Weyl) fermions are more fundamental than Dirac fermions, it is natural to interpret the axial anomaly as due to the anomaly of the chiral current,

$$J_A^\mu = J_L^\mu - J_R^\mu, \tag{6.36}$$

$$J_L^\mu = i\psi\gamma^\mu\frac{1-\gamma^5}{2}\psi; \qquad J_R^\mu = i\psi\gamma^\mu\frac{1+\gamma^5}{2}\psi. \tag{6.37}$$

The derivation is almost identical to the case of the axial anomaly: the result is

$$\partial_\mu J_L^\mu = \frac{g^2}{32\pi^2}F_{\mu\nu}^a\tilde{F}^{\mu\nu\,a} \tag{6.38}$$

for one lefthanded fermion.

The crucial generalization is the nonabelian anomaly. Consider a theory with a chiral fermion, with a (continuous) global symmetry G_F. Suppose that the group G_F is "gauged", by coupling to external gauge fields A:

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,e^{-\int\,dx\,\bar{\psi}i\gamma\mathcal{D}^A\psi}. \tag{6.39}$$

Now, under the gauge transformation

$$A_\mu \to A_\mu - \mathcal{D}_\mu v, \qquad \mathcal{D}_\mu v = \partial_\mu v + ig[A_\mu, v], \tag{6.40}$$

$$\Gamma(A) \to \Gamma(A) + \int dx\, v^a\, \mathcal{D}_\mu \frac{\delta \Gamma(A)}{\delta\, A^a_\mu}. \tag{6.41}$$

If the theory is invariant under G_F one should have

$$\frac{\delta \Gamma(A)}{\delta\, v^a} = 0, \quad \therefore \quad \mathcal{D}_\mu \langle J^a_\mu \rangle = 0, \quad J^a_\mu = \frac{\delta \Gamma(A)}{\delta\, A^a_\mu}. \tag{6.42}$$

Actually, the variation is nonvanishing and is given by

$$\begin{aligned} \delta_{v_L} \Gamma(A) &= \frac{1}{24\pi^2} \int d^4x\, \mathrm{Tr}\, v(x) \epsilon^{\lambda\mu\alpha\beta}\, \partial_\lambda \left(A_\mu \partial_\alpha A_\beta + \frac{1}{2} A_\mu A_\alpha A_\beta\right) \\ &= \frac{1}{24\pi^2} \int d^4x \int \mathrm{Tr}\, v(x)\, d(AdA + \frac{1}{2} A^3). \end{aligned} \tag{6.43}$$

Note that it is proportional to the group constant

$$\mathrm{Tr} T^a \{T^a T^c\} = d^{abc}(\underline{r}) \tag{6.44}$$

which depends on the representation to which the fermions ψ belong. In particular, the ratio $d^{abc}(\underline{r})/d^{abc}(\underline{r}_0)$ where $\underline{r}_0$ is the fundamental representation, depends only on $\underline{r}$. For $SU(N)$ see Appendix.

The fact that the anomaly (6.43) emerges as a variation of the effective action $\Gamma(A)$ implies a consistency condition (Wess-Zumino [**?**]),

$$(\delta_{v_1} \cdot \delta_{v_2} - \delta_{v_2} \cdot \delta_{v_1})\, \Gamma(A) = \delta_{[v_1, v_2]}\, \Gamma(A), \tag{6.45}$$

and this fixes the anomalous part of $\Gamma(A)$ modulo normalization, as the integral of the anomaly.

As the low-energy effective action in QCD contains pions and kaons (pseudo-Nambu-Golstone bosons) coupled to the external $G_F = SU(N_f \times SU(N_f) \times U_V(1)$ gauge bosons (the physical photon, W, Z bosons couple to a subgroup of G_F), the consequence of the anomaly involves precise physical prediction on certain hadronic processes involving weak gauge bosons ($\pi_0 \to 2\gamma$ discussed above being an example). Actually, there are some purely hadronic process due to the anomaly.

Witten [**?**] observed that the integrated anomaly - the so-called Wess-Zumiino effective action - can be written as a boundary term on $S^4 \sim R^4$ of R^5. For $N_f = 3$, the integrated anomaly takes the form,

$$\Gamma_{eff} = n \int d\Sigma_{i_1 i_2 \ldots i_5}\, \omega_{i_1 i_2 \ldots i_5} \equiv n\, \Gamma(U), \tag{6.46}$$

$$\omega_{i_1 i_2 \dots i_5} = -\frac{i}{240\,\pi^2}\,\mathrm{Tr}\,(U^{-1}\partial_{i_1}U)(U^{-1}\partial_{i_2}U)\dots(U^{-1}\partial_{i_5}U), \tag{6.47}$$

where $U(x^\mu, y) = \exp\left[\frac{2i}{F_\pi}\,\Sigma\lambda^a\,\pi^a(x^\mu, y)\right]$ represents the Goldstone modes of $\frac{SU(3)\times SU(3)}{SU(3)} \sim SU(3)$ suitably continued to R^5. As

$$\pi_5(SU(3)) \sim \mathbb{Z}, \tag{6.48}$$

the coefficcient n turns out to be quantized to integer values. The actual Wess-Zumino effective action of QCD contains $n = N_c = 3$, but the fact that n should be *a priori* an integer, is highly non-trivial.

In the presence of external gauge fields coupled to the global symmetry, the latter must be "gauged", and the Wess-Zumino-Witten action must be made gauge invariant under the anomaly-free part of local G_F (in this case the vectorial $SU(3)$): the result is

$$\tilde{\Gamma}(U, A_\mu) = \Gamma(U) + \frac{1}{48\pi^2}\int d^4x\,\epsilon^{\mu\nu\alpha\beta}\,Z_{\mu\nu\alpha\beta}, \tag{6.49}$$

where $(U_{\nu L} = (\partial_\nu U)\,U^{-1},\ U_{\nu R} = U^{-1}\partial_\nu U)\,)$,

$$Z_{\mu\nu\alpha\beta} = -\mathrm{Tr}[\,A_{\mu L}U_{\nu L}U_{\alpha L}U_{\beta L} + (L \leftrightarrow R)\,] + \dots \tag{6.50}$$

given in Eq.(24) of Witten, NPB223(1983) 422. This effective action reproduces the anomalous variations under $SU_A(3)$ calculated in the underlying theory.

6.8. Cancellation of gauge anomalies in the Standard model

As the Glashow-Weinberg-Salam theory is chiral, the anomaly discussed above is potentially dangerous for the consistency of $SU(3) \times SU_L(2) \times U_Y(1)$ gauge theory: the presence of the anomaly implies the failure of gauge invariance. In the satndard model with quarks and leptons, these anomalies cancel out completely, due to the particular charges these known fermions carry.

6.9. 't Hooft's anomaly matching conditions

The nonabelian anomaly implies very nontrivial consistency condition on possible realization of symmetries at low energies. The anomaly represents the way the system responds to the variation of the external gauge fields A_μ^a in G_F (whether or not they represent actual, physical particles): the answer must be the same whether the action of the fundamental system or the effective action valid at low energies, is used to calculate it.

In the first case, the quantum loop calculation gives the anomaly discussed in the previous subsection. If instead one uses an effective low-energy action, which takes into account all quantum effects integrated down to the infrared scale considered and which describe the low-energy degrees of freedom such as the Nambu-Goldstone bosons or composite massless fermions. If the relevant degrees of freedom involves only the latter (symmetry broken spontaneously), then the consequence is the $\pi_0 \to 2\gamma$ and other anomalous amplitudes contained in the Wess-Zumino- Witten action (6.47).

Suppose instead that a global symmetry G_F in a strongly interacting theory (with guge group G_S), remains unbroken. *In such a case, the low-energy effective theory must contain massless, composite fermions, so that the loops of these massless "baryons" reproduce exactly the anomaly of the underlying theory.* See Fig. . As in general composite fermions, being singlets of G_S, are in different representations than those of the fundamental fermions, such an agreement involves very nontrivial, algebraic relations about their numbers and charges. Thus if the theory confines, without breaking the global symmetry, these "'t Hooft anomaly matching conditions" imposes a stringent constraints on possible dynamical outcome of the theory.

N. B The idea behind the 't Hooft's conditions is thus essentially the same as that led to the determination of the $\pi_0 \to 2\gamma$ amplitude. The anomaly of the system must be carried either by massless Nambu-Goldstone bosons or by massless composite fermion loops. Both produce - effectively - a pole at $q^2 = 0$ in the amplitude (Fig.)

N. B Even if the theory does not confine, if the theory approach a nontrivial fixed-point in the infrared, any alternative description of the system must involve the set of fermions with an appropriate amount of anomaly. This constraint was used by Seiberg in his discovery of the conformal Window in SQCD.

N. B In a supersymmetric strongly interacting theory, with a spontaneouly broken global symmetry G_F, the anomaly matching is subtler, as the fermionic partners of the Nambu-Goldstone bosons are massless, and carry nontrivial charges. In the calculation of the "pion decay amplitude" in such a context, the contribution of the loops due to the low-enrgy fermions must be subtracted from the fundamental anomaly.

6.10. Fujikawa's derivation of anomalies

Consider

$$W = \int \mathcal{D}\psi \, \bar{\mathcal{D}}\psi \, e^{-\int d^4 \bar{\psi} \, i\gamma \mathcal{D}^A \psi} \tag{6.51}$$

$$\mathcal{D}\psi \, \bar{\mathcal{D}}\psi \; = \prod da_n d\bar{b}_n, \tag{6.52}$$

where

$$i\gamma \mathcal{D}^A \, \psi_n = \lambda_n \psi_n, \tag{6.53}$$

and

$$\psi = \sum_n a_n \psi_n; \qquad \bar{\psi} = \sum_n \bar{b}_n \psi_n^\dagger. \tag{6.54}$$

Let us consider the change of variable,

$$\psi \to e^{i\alpha(x)} \gamma_5 \, \psi, \qquad \bar{\psi} \to \bar{\psi} \, e^{i\alpha(x)} \gamma_5, \tag{6.55}$$

which has the form of a $U_A(1)$ transformation. Namely

$$\psi = \psi' + i\alpha(x)\gamma_5\psi'; \qquad \bar{\psi} = \bar{\psi}' + i\alpha(x)\bar{\psi}'\gamma_5. \tag{6.56}$$

As

$$\psi = \sum_n a_n \psi_n = \sum_n a_n \, e^{i\alpha(x)} \gamma_5 \psi_n' = \sum a_n' \psi_n, \tag{6.57}$$

$$da_m' = \sum_n [\int dx \, \psi_m^* \, (1 + \alpha(x) \, \gamma_5 \,) \, \psi_n \,] \, da_n \equiv \sum_n C_{mn} \, da_n; \tag{6.58}$$

$$\prod_m da_m' = \det C \prod da_n, \qquad \prod da_n = (\det C)^{-1} \prod_m da_m' \tag{6.59}$$

and analogously

$$\prod_n d\bar{b}_n = \prod_n d\bar{b}_n' \, (\det C)^{-1}. \tag{6.60}$$

The functional Jacobian is $J = (\det C)^{-2}$. By using (for a matrix of the form $C = 1 + A$),

$$\det C = e^{\log \det C} = e^{\mathrm{Tr} \log(1+A)} \simeq e^{\mathrm{Tr} A}, \tag{6.61}$$

$$J = \exp[-2i \int dx\, \alpha(x) \sum_n \psi_n^*(x)\gamma_5\psi_{n(x)},]. \qquad (6.62)$$

Naïvely,

$$\sum_n \psi_n^*(x)\, \gamma_5\, \psi_{n(x)} = \mathrm{Tr}\gamma_5 = 0, \qquad (6.63)$$

but actually it is an indefinite sum of the form, $0 \cdot \infty$. In order to define better the sum, we introduce the regulator factor following Fujikawa,

$$\begin{aligned} \sum_n \psi_n^*(x)\gamma_5\psi_{n(x)} &= \lim_{M\to\infty} \sum_n \psi_n^*(x)\, \gamma_5\, e^{-\lambda_n^2/M^2}\, \psi_{n(x)} \\ &= \lim_{M\to\infty} \sum_n \psi_n^*(x)\, \gamma_5\, e^{(\gamma\cdot\mathcal{D})^2/M^2} \psi_{n(x)} = \lim_{M\to\infty} \mathrm{Tr}\, \gamma_5 e^{(\gamma\cdot\mathcal{D})^2/M^2}. \qquad (6.64) \end{aligned}$$

The sum, now well defined, can be evaluated in any basis, e.g., in plane wave basis,

$$\mathrm{Tr}\,(\cdots) \to \int \frac{d^4p}{(2\pi)^4}\, e^{ipx}\, \mathrm{tr}\,(\cdots)e^{-ipx} \qquad (6.65)$$

where $\mathrm{tr}\,(\cdots)$ stands for the spin trace. Now

$$(\gamma\cdot\mathcal{D})^2 = D_\mu^2 + \sigma_{\mu\nu}\, F^{\mu\nu}, \qquad \sigma_{\mu\nu} = \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{4}, \qquad (6.66)$$

$$D_\mu = \partial_\mu - i\, g\, A_\mu = iP_\mu - i\, g\, A_\mu = i\,(P - gA), \qquad P_\mu = -i\partial_\mu, \qquad (6.67)$$

and

$$D_\mu^2 = -(P - igA)^2 = -P_\mu^2 + g(P_\mu A_\mu + A_\mu P_\mu) - g^2 A_\mu^2 \equiv P^2 - \Delta^2, \qquad (6.68)$$

$$\Delta = -g(P_\mu A_\mu + A_\mu P_\mu) + g^2 A_\mu^2. \qquad (6.69)$$

Thus

$$\mathrm{Tr}\, \gamma_5\, e^{(\gamma\cdot\mathcal{D})^2/M^2} = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\, \mathrm{tr}\,[\gamma_5\, e^{-P^2-\Delta+\sigma_{\mu\nu}\, F^{\mu\nu}}\,]\, e^{-ipx}. \qquad (6.70)$$

We now expand in the "small" operators $(-\Delta + \sigma_{\mu\nu}\, F^{\mu\nu})/M^2$, as $P^2 \sim O(M^2)$. The three operators in the exponent do not commute; one can however use the formula

$$\begin{aligned} e^{A+B} &= e^A + \int_0^1 d\alpha\, e^{\alpha A}\, B\, e^{(1-\alpha)A} + \\ &+ \int \prod_{i=1}^3 d\alpha_i\, \delta(1 - \sum \alpha)\, e^{\alpha_1 A}\, B\, e^{\alpha_2 A}\, B\, e^{\alpha_3 A} + \dots. \qquad (6.71) \end{aligned}$$

As

$$\int \frac{d^4p}{(2\pi)^4} \sim O(M^4), \tag{6.72}$$

the terms higher than quadratic in $(-\Delta + \sigma_{\mu\nu} F^{\mu\nu})/M^2$ are negligible in the limit $M \to \infty$. On the other hand, the spin trace $\mathrm{tr}\gamma_5 \cdots]$ vanishes unless at least four γ matrices appear. It follows that the only term surviving in the limit $M \to \infty$ is the quadratic term in $\sigma_{\mu\nu} F^{\mu\nu})/M^2$. This can be easily evaluated as

$$\begin{aligned}
& \frac{1}{2}\mathrm{tr}(\gamma_5 \sigma_{\mu\nu}\, \sigma_{\rho\sigma} \, \frac{F^{\mu\nu}F^{\rho\sigma}}{M^4} \int \frac{d^4p}{(2\pi)^4} e^{-p^2/M^2} \\
= \; & \frac{1}{2}\mathrm{Tr}\frac{F_{\mu\nu}F_{\rho\sigma}}{M^4}\frac{4}{16}\mathrm{tr}(\gamma_5\gamma_\mu\, \gamma_\nu\, \gamma_\rho\, \gamma_\sigma) \cdot \frac{1}{(2\pi)^4} \cdot \frac{\pi^2}{\Gamma(2)} \cdot \int d(p^2)\, p^2 e^{-p^2/M^2} \\
= \; & \mathrm{Tr}\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{1}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}.
\end{aligned} \tag{6.73}$$

By introducing appropriate source termsand by considering the change of variables

6.11. Konishi anomaly

In supersymmetric gauge theories, the chiral $U(1)$ anomaly associated with each matter field is promoted to supersymmetric multiplet of anomalies. In a theory with chiral superfields Φ_i and superpotential $\mathcal{P}(\Phi)$, it reads

$$-\frac{\bar{D}^2}{4}(\Phi^\dagger{}_i e^V \Phi_i) = \Phi_i \frac{\delta \mathcal{P}}{\delta \Phi_i} + C_{\Phi_i}\left(\frac{g^2}{32\pi^2}\right) WW \tag{6.74}$$

where C_{Φ_i} is the quadratic Casimir of the gauge group associated with the representation to which Φ_i belongs. The first term on the r.h.s. is the normal field variation and the second term is the anomaly. Written this way this can be seen as the anomalous equation of motion. The point is that this relation holds exactly as a relation among the chiral ring of operators and hence among the VEV of the terms involved.

Eq.(6.74) can be derived in various ways, by using (supersymmetric) point-splitting method, supergraph 1 loop calculation, BPHZ regularization, Pauli-Villars regularization, and in the functional (à la Fujikawa) method.[2] We discuss below this last method, as it is easiest then to generalize such identity to much

[2]The proof has been given by using these techniques also in the component formalism (Konishi).

more general identities. Also this method can be used in chiral gauge theories while some method (e.g., Pauli-Villars) requires the model to be vectorlike. Let's study a supersymmetric gauge theory with generic field content $\{\Phi_i\}$. Consider the functional integral,

$$e^{Z[J]} = \int \mathcal{D}\Phi \ldots \exp[i(S_{matter} + S_{gauge} + S_{Sources}], \tag{6.75}$$

$$S_{matter} = \sum_i \int d^8z[\,\Phi^\dagger{}_i e^V \Phi_i] + \int d^6z\, \mathcal{P}(\Phi) + h.c. \tag{6.76}$$

$$S_{gauge} = \frac{1}{4}\int d^6z \mathrm{Tr} WW \tag{6.77}$$

and $S_{Sources}$ is a generic source term. For SQCD $\{\Phi_i\} = \{Q, \tilde{Q}\}$. Let us now consider the change of variable for one of the chiral superfield

$$U(1)_\Phi \quad : \qquad \Phi \to e^{iA(z)}\Phi \tag{6.78}$$

with other variables kept fixed, where $A(z)$ is a generic chiral superfield. This introduces

$$\delta S_{matter} = \int d^8z\, \Phi^\dagger e^V iA(z)\Phi + \int d^6z\, iA(z)\Phi\frac{\delta\mathcal{P}}{\delta\Phi} \tag{6.79}$$

(for SQCD the variation of the superpotential is $m\tilde{\Phi}iA(z)\Phi$) Naively one would have

$$\frac{\delta S}{\delta A} = 0 = -\frac{\bar{D}^2}{4}(\Phi^\dagger e^V \Phi) + \Phi\frac{\delta\mathcal{P}}{\delta\Phi} + \delta S_{Sources} \tag{6.80}$$

but actually one must also take into account the anomalous jacobian of the transformation 6.78, which is given by

$$\mathcal{J}(\Phi'/\Phi) = \det_c \mid \delta\Phi'_{z'}/\delta\Phi_z \mid = \det_c \langle z' \mid \exp\left\{iA\left(-\frac{\bar{D}^2}{4}\right)\right\} \mid z\rangle$$
$$= \exp\left\{\mathrm{Tr}_c\left[iA\left(-\frac{\bar{D}^2}{4}\right)\right]\right\} \tag{6.81}$$

where the subscript "c" stays for "chiral measure", that is $d^6z = d^4xd^2\theta$. Note that, being the delta function in the superspace $\langle z' \mid z\rangle = \delta^8(z-z')$, the correct delta function for the chiral measure is $\langle z' \mid -\frac{\bar{D}^2}{4} \mid z\rangle = \delta^6(z-z') = \delta\Phi_{z'}/\delta\Phi_z$. Apparently the exponent in 6.81 vanishes because $\delta^6(0) = 0$, but actualyy it is an infinite sum and it needs to be regularized. For this purpose we adopt a generalization of the Fujikawa method and write the trace as

$$\lim_{M\to\infty} \mathrm{Tr}_c\left[i\,A\,e^{\frac{L}{M^2}}\left(-\frac{\bar{D}^2}{4}\right)\right] \tag{6.82}$$

$$L = \frac{1}{16}\bar{D}^2 e^{-V} D^2 e^V = \not{D}^2 + \cdots \tag{6.83}$$

The choice of L is justified by the fact that it is the simplest operator with the following remarkable properties

1. L is manifestly supersymmetric
2. L is manifestly chiral
3. L is gauge-covariant, in particular it transforms

$$L \to e^{-iA} L e^{iA} \tag{6.84}$$

4. L contains $\not{D}^2$ as a component (see the original Fujikawa method)

Let's evaluate the trace in 6.82, remembering that $\bar{D}^2 = 0$

$$e^{\frac{L}{M^2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{L}{M^2}^n \quad ; \qquad 1_{--} \equiv -\frac{\bar{D}^2}{4} \tag{6.85}$$

$$\begin{aligned} L1_{--} &= \frac{1}{16}[\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} e^{-V} D^\alpha D_\alpha e^V]1_{--} \\ &= \frac{1}{16}[\bar{D}^2 e^{-V}(D^2 e^V) + 2\bar{D}^2 e^{-V}(D_\alpha e^V)D^\alpha + \bar{D}^2 D^2]1_{--} \end{aligned} \tag{6.86}$$

$$\begin{aligned} \bar{D}^2 D^2 1_{--} &= \{\bar{D}_{\dot\alpha}[\bar{D}^{\dot\alpha}, D^2] + [\bar{D}_{\dot\alpha}, D^2]\bar{D}^{\dot\alpha}\}1_{--} \\ &= \bar{D}_{\dot\alpha}(D^\alpha\{\bar{D}^{\dot\alpha}, D_\alpha\} - \{\bar{D}^{\dot\alpha}, D^\alpha\}D_\alpha)1_{--} \\ &= 2\bar\sigma^{\mu\dot\alpha\alpha}p_\mu(2\{\bar{D}_{\dot\alpha}, D_\alpha\}) \\ &= 8\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\alpha}p_\mu p^\nu 1_{--} = 16 g^{\mu\nu}p_\mu p^\nu 1_{--} = 16p^2 1_{--} \end{aligned} \tag{6.87}$$

$$\bar{D}^2 e^{-V}(D_\alpha e^V)D^\alpha 1_{--} = -4W^\alpha D_\alpha + 8C^\mu p_\mu \tag{6.88}$$

$$C^\mu \equiv -\frac{1}{2}\sigma^\mu_{\alpha\dot\alpha}(\bar{D}^{\dot\alpha} e^{-V} D^\alpha e^V) \tag{6.89}$$

$$L1_{--} = (p^2 - \frac{1}{2}W^\alpha D_\alpha + C^\mu p_\mu + F)\,1_{--}; \tag{6.90}$$

$$F \equiv \frac{1}{16}(\bar{D}^2 e^{-V} D^2 e^V). \tag{6.91}$$

To select the non-vanishing part of 6.82 we need 2 covariant derivatives from the expansion of L; indeed

$$\langle \theta, \bar{\theta} \mid D_\alpha D_\beta \bar{D}^2 \mid \theta, \theta \rangle = 8\epsilon_{\alpha\beta} \tag{6.92}$$

while a different number of D's would yield zero. Furthermore

$$\begin{aligned} \langle x \mid \exp\left[\frac{p^2 + \mathcal{O}(p)}{M^2}\right] \mid x \rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-\frac{k^2}{M^2}} \left(1 + \mathcal{O}\left(\frac{1}{M}\right)\right) \\ &= \frac{i}{16\pi^2} M^4 + \mathcal{O}(M^3) \end{aligned} \tag{6.93}$$

so we need just the second order in $W^\alpha D_\alpha$ and after a rescaling $V \to 2gV$ we are led to

$$\lim_{M\to\infty} \mathrm{Tr}_c \left[iAe^{\frac{L}{M^2}} \left(-\frac{\bar{D}^2}{4}\right)\right] = \int d^6z iA(z) \left(\frac{g^2}{32\pi^2}\right) W^\alpha W_\alpha \tag{6.94}$$

We remark that the result is stable respect to the choice of the regulator; a generic function $f(L/M^2)$ such that $f(\infty) = f'(\infty) = \ldots = 0$, $f(0) = 1$ would be a good cutoff.
The correct relation that follows from the invariance of the action is then

$$-\frac{\bar{D}^2}{4}(\Phi^\dagger e^V \Phi) + \Phi \frac{\delta \mathcal{P}}{\delta \Phi} + \left(\frac{g^2}{32\pi^2}\right) W^\alpha W_\alpha + \delta S_{Sources} = 0 \tag{6.95}$$

Actually the whole derivation of the anomalous WT identities goes through almost unmodified if $A(z)$ is considered to be an arbitrary chiral functions of Φ's and W^α's,

$$A = f(\Phi, W). \tag{6.96}$$

A systematic study of the complete set of relations and the solution of those, has recently been given by Cachazo, Douglas, Seiberg and Witten.

7. Instantons in gauge field theories

The Lagrangian for a gauge theory is

$$L = \bar{\psi}_k \gamma^\mu (\partial_\mu - igA_\mu)\psi_k - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \tag{7.1}$$

where $G^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + gf^{abc} A^b_\mu A^c_\nu$.

We are interested in the Yang-Mills Euclidean action

$$S_E = \frac{1}{4}\int d^4x\, F^{a2}_{\mu\nu} = \frac{1}{8}\int d^4x\, [(F^A_{\mu\nu} \mp \tilde{F}^a_{\mu\nu})^2 \pm 2F^{a\mu\nu}\tilde{F}^a_{\mu\nu}] \tag{7.2}$$

where $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{a\alpha\beta}$ is the 'dual' of $F^a_{\mu\nu}$. The second term

$$\mathcal{P} = \frac{1}{4}\int d^4x F^{a\mu\nu}\tilde{F}^a_{\mu\nu} \tag{7.3}$$

in the Euclidean action is a topological invariant, because $F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = \partial_\mu \mathcal{C}^\mu$ is a total derivative. $\mathcal{P}$ is called (up to a constant factor) the Pontryagin invariant and

$$\mathcal{C}^\mu = \varepsilon^{\mu\nu\alpha\beta}(A^a_\nu \partial_\alpha A^a_\beta + \frac{2}{3}g f^{abc} A^a_\nu A^b_\alpha A^c_\beta) \tag{7.4}$$

is called a Chern-Simons secondary characteristic class term.

If we look for solutions that minimize the Euclidean action for a certain value of the Pontryagin invariant, we must minimize only the first term of equation 7.2, *i.e.* we must solve the equation

$$F^a_{\mu\nu} = \pm\tilde{F}^a_{\mu\nu} \tag{7.5}$$

These solutions are called self-dual and antiself-dual, respectively. To construct some explicit solutions for a $SU(2)$ gauge group, we introduce the equivalent of σ-matrices in an Euclidean space:

$$\tau^\mu = (1, i\sigma) \qquad \bar{\tau} = (1, -i\sigma) \tag{7.6}$$

$$\tau_{\mu\nu} = \frac{1}{4}(\tau_\mu\bar{\tau}_\nu - \tau_\nu\bar{\tau}_\mu) \qquad \bar{\tau}_{\mu\nu} = \frac{1}{4}(\bar{\tau}_\mu\tau_\nu - \bar{\tau}_\nu\tau_\mu) \tag{7.7}$$

Now an explicit solution is

$$A^{INST}_\mu = \frac{T^a}{2}A^a_\mu = -\frac{2i}{g}\frac{\tau_{\mu\nu}(x-x_0)^\nu}{(x-x_0)^2+\rho^2} \tag{7.8}$$

where x_0 is the center of the instanton and ρ is its size. $\tau_{\mu\nu}$ is selfdual and so the entire solution, in fact

$$F^{INST}_{\mu\nu} = 4\rho^2 f^2(x)\tau_{\mu\nu} \qquad f(x) = \frac{1}{(x-x_0)^2+\rho^2} \tag{7.9}$$

The corresponding antiself-dual solution is

$$A^{ANTIINST}_\mu = -\frac{2i}{g}\frac{\bar{\tau}_{\mu\nu}(x-x_0)^\nu}{(x-x_0)^2+\rho^2} \tag{7.10}$$

We can calculate now

$$S_E^{INST} = \frac{1}{2g^2}\int d^4x\,\mathrm{Tr}F_{\mu\nu}^2 = \frac{8\pi^2}{g^2} \tag{7.11}$$

and the factor $e^{-S_E} = e^{-\frac{8\pi^2}{g^2}}$ that appear in the functional integral is the non-perturbative tunnelling instantonic factor.

Note that we have

$$\mathrm{Tr}\frac{1}{16\pi^2}F\tilde{F} = \partial_\mu \mathcal{K}^\mu \qquad \mathcal{K}^\mu = \frac{1}{16\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\mathrm{Tr}(F_{\alpha\beta}A_\gamma - \frac{2}{3}A_\alpha A_\beta A_\gamma) \tag{7.12}$$

so one can construct a topological invariant

$$q = \int d^4x\,\mathrm{Tr}\frac{1}{16\pi^2}F\tilde{F} = \int d^4x\,\partial_\mu\mathcal{K}^\mu \tag{7.13}$$

and if A_μ tends to a pure gauge field in all directions

$$A_\mu \longrightarrow iU^{-1}\partial_\mu U \quad \text{for\_}|x| \to +\infty \tag{7.14}$$

then q can be expressed as

$$q = \frac{1}{24\pi^2}\int dS^\mu\,\varepsilon_{\mu\nu\alpha\beta}\mathrm{Tr}[(U^{-1}\partial^\nu U)(U^{-1}\partial^\alpha U)(U^{-1}\partial^\beta U)] \tag{7.15}$$

which is an integer: it is the winding number of the function which goes from the boundary of $\mathbb{R}^4$ to $SU(2)$, so it is topologically equivalent to a function $S^3 \to S^3$, and $\Pi_3(S^3) = \mathbb{Z}$. As a geometric example of this behaviour we can consider a vector field $n^a(x) : S^2 \to S^2$, $[n^a(x)]^2 = 1$:

$$N = \frac{1}{4\pi}\int_{\partial V}\varepsilon^{abc}n^a(x)dn^b dn^c = \Pi_2(SO(3)) = \mathbb{Z} \tag{7.16}$$

Back to physics, we can write these quantities for the instanton solution above:

$$U = \frac{\tau_\mu x_\mu}{\sqrt{x^2}} \qquad U\partial_\mu U^\dagger = -\frac{2}{x^2}\tau_{\mu\nu}x^\nu \tag{7.17}$$

8. Instanton calculus

We are interested in the Euclidean path integral

$$\int \mathcal{D}A\,e^{-S} \qquad A_\mu = A_\mu^{INST} + Q_\mu \tag{8.1}$$

around the local instanton minimum of the action instead of around the vacuum solution. We find immediately

$$\int \mathcal{D}A\, e^{-S} = \int \mathcal{D}Q\, e^{-(S_{cl}+QMQ+\dots)} = e^{-S_{cl}}(\mathrm{Det}M)^{-\frac{1}{2}} \tag{8.2}$$

This kind of contribution to the functional integral exists in every gauge theory with a gauge group G which contains $SU(2)$ as a subgroup.

Note that $S_{classical}$ does not depend on some parameters of the instanton, like x_0, ρ, or the gauge $SU(2)/G$. This means that there are 'zero modes'. We can treat this problem with a bosonic field B as an example. We can write

$$B = B^{cl} + B^q \qquad S = S_{cl} + \frac{1}{2}B^q M B^q + \dots \tag{8.3}$$

then we can find a complete orthogonal set of eigenfunctions for M:

$$M\chi_i = \varepsilon_i \chi_i \qquad \langle \chi_i|\chi_j\rangle = \int d^4x\, \chi_i^*(x)\chi_j(x) = u_i\delta_{ij} \tag{8.4}$$

and we can expand B^q as a linear superposition with functional coefficients ξ_i

$$B^q = \sum_i \xi_i \chi_i(x) \tag{8.5}$$

Now we consider a parameter γ which appears in $B^{cl} = B^{cl}(\gamma)$, and suppose the action doesn't depend on it; then there is a mode with zero eigenvalue

$$\chi_0 = \frac{\partial B^{cl}}{\partial \gamma} \qquad \varepsilon_0 = 0 \tag{8.6}$$

Then the functional integral is

$$\int \mathcal{D}B = \int \mathcal{D}B^q = \int \prod \left(\frac{u_i}{2\pi}\right)^{\frac{1}{2}} d\xi_i = \int \left(\frac{u_0}{2\pi}\right)^{\frac{1}{2}} d\xi_0 \prod{}' \left(\frac{u_i}{2\pi}\right)^{\frac{1}{2}} d\xi_i \tag{8.7}$$

where the product in the last tern is taken without the zero mode. We know that $M\chi_0 = 0$, so

$$S = S^{cl} + \frac{1}{2}\sum{}' \varepsilon_i u_i \xi_i^2 + \dots \tag{8.8}$$

$$\int \mathcal{D}B\, e^{-S} = e^{-S^{cl}} \int \left(\frac{u_0}{2\pi}\right)^{\frac{1}{2}} d\xi_0\, (\mathrm{Det}'M)^{-\frac{1}{2}} \tag{8.9}$$

Now we use the fact that $B(\gamma) \approx B(\gamma_0) + \frac{\partial B^{cl}}{\partial \gamma}\triangle\gamma = B(\gamma_0) + \chi_0(\gamma_0)\triangle\gamma$ and insert the identity

$$1 = u_0 \int d\gamma\, \delta(\langle B - B^{cl}(\gamma_0)|\chi_0(\gamma_0)\rangle) = u_0 \int d\gamma\, \delta(u_0\xi_0) \tag{8.10}$$

in the functional integral. Finally we obtain

$$\int \mathcal{D}B\, e^{-S} = \int d\gamma \left(\frac{u_0}{2\pi}\right)^{\frac{1}{2}} e^{-S^{cl}} (\mathrm{Det}' M)^{-\frac{1}{2}} = \left(\frac{u_0}{2\pi}\right)^{\frac{1}{2}} e^{-S^{cl}} \frac{1}{(\prod' \varepsilon_i)^{\frac{1}{2}}} \Gamma \quad (8.11)$$

where $\Gamma = \int d\gamma$.

In the case of the instanton previously considered, there are

- 4 zero modes for x_0^μ;
- 1 zero mode for ρ;
- 3 zero modes due to rotations;
- $4N$-8 modes due to gauge choice (called 'gauge zero modes') for the gauge group $SU(N)$.

There are some fine points related to transformations and gauge choices (t'Hooft, Bernard). The result is

$$\int \mathcal{D}A\, e^{-S} = \frac{4}{\pi^2} \frac{(4\pi^2)^{2N}}{(N-1)!(N-2)!} \int d^4x \frac{d\rho}{\rho^5} \left(\frac{4\pi^2}{g}\right)^{2N} (\rho\mu)^{4N} e^{-\frac{8\pi^2}{g^2}} \int \mathcal{D}A'\, e^{-(S-S^{cl})} \quad (8.12)$$

It is possible to repeat these passages for fermionic fields, *i.e.* in SQCD Lagrangian

$$L = i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi + \sqrt{2} i g \lambda \psi \phi^* + h.c. \quad (8.13)$$

Now we define

$$\psi = \sum a_m \eta_m(x) \qquad \bar{\psi} = \sum \bar{b}_n \bar{\xi}_n^*(x) \quad (8.14)$$

$$\bar{D}\eta_m = \bar{k}_m \bar{\xi}_m \qquad D\bar{\xi}_n = k_n \eta_n \quad , \quad \bar{D} = \bar{\tau}^\mu (\partial_\mu - igA_\mu) \quad (8.15)$$

$$S_{mat} = \sum a_n \bar{b}_m \int d^4x\, \bar{\xi}_m^* \xi_n \bar{k}_n = \sum a_n \bar{b}_n u_n \bar{k}_n \quad (8.16)$$

where $\int d^4x\, \bar{\xi}_m^* \xi_n = u_n \delta_{mn}$. There is a zero mode $\bar{k}_0 = 0$, so

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi} = \int \prod_m da_m\, u_m^{-\frac{1}{2}} \prod_n d\bar{b}_n\, u_n^{-\frac{1}{2}} \quad (8.17)$$

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{-S} = \int da_0\, u_0^{-\frac{1}{2}} \prod{}' \bar{k}_n = \int da_0\, u_0^{-\frac{1}{2}} \mathrm{Det}' \bar{D} \quad (8.18)$$

because for fermionic integrals $\int da_n\, d\bar{b}_n\, e^{-a_n \bar{b}_n c} = c$. In QCD and SQCD the quark fields have one zero mode; in SYM there are gluinos λ in the adjoint representation, so there are $2N_C$ zero modes, where $2N_C$ in the Dinkin index of the anomaly $\partial_\mu J^\mu_\lambda = (2N_C)\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$. For $N_C = 2$, a zero modes is

$$\psi^0_{\alpha,r} = \frac{1}{\pi}\varepsilon_{\alpha r}\rho f(x)^{\frac{3}{2}} \tag{8.19}$$

where α refers to spin degrees of freedom, while r refers to color. This mode solves $\bar{\sigma}^{\mu\nu}D_\nu\psi_0 = 0$; for $|x| \to \infty$ goes $\sim x^{-\frac{3}{2}}$ and it is possible to find a gauge (called singular gauge) so that this solution goes asimptotically as the fermionic propagator S_F. The singular gauge transformation and the corresponding instanton solution are

$$U = \frac{(x-x_0)_\mu \tau^\mu}{\sqrt{(x-x_0)}} \qquad A^{INST}_\mu = \frac{2i\tau_{\mu\nu}(x-x_0)^\nu}{(x-x_0)^2+\rho^2} \tag{8.20}$$

To see a useful application of instanton calculus, consider the following VEV in QCD, which is not invariant under axial $U(1)$ transformation:

$$\langle 0|T(\psi_{i_1}(x_1)\psi_{i_2}(x_2)\ldots\tilde{\psi}_{j_{N_f-1}}(y_{N_f-1})\tilde{\psi}_{j_{N_f}}(y_{N_f}))|0\rangle \tag{8.21}$$

The idea is that $\int da_{0\,i}$ is zero unless there is an a_0 contribution for every field. The result is

$$\sim const\,\varepsilon_{i1\ldots i_{N_f}}\varepsilon_{j1\ldots j_{N_f}}\int d^4x_0\,\psi_{0\,i_1}(x_1)\psi_{0\,i_2}(x_2)\ldots\tilde{\psi}_{0\,j_{N_f-1}}(y_{N_f-1})\tilde{\psi}_{0\,j_{N_f}}(y_{N_f}) \tag{8.22}$$

and in singular gauge it becomes

$$\sim const \prod_{i,j} S_F(x_i - x_0)S_F(y_j - y_0) \tag{8.23}$$

t'Hooft recognized in this result a generation of instantonic L_{EFF},

$$L_{EFF} = \text{const.}\,\psi_{i_1}(x_1)\psi_{i_2}(x_2)\ldots\tilde{\psi}_{j_{N_f-1}}(y_{N_f-1})\tilde{\psi}_{j_{N_f}}(y_{N_f}) + h.c. \tag{8.24}$$

Unfortunately the constant factor

$$\text{const.} = \int_0^\infty d\rho\,\rho^{\frac{11N_C-2N_f}{3}}e^{-\frac{8\pi}{g_0^2}} \tag{8.25}$$

is infrared divergent. Although in the actual world there is a cutoff of order of $\frac{1}{\Lambda_C}$ in the integration over ρ due to confinement, this makes a quantitative calculation of L_{EFF} more difficult.

In any event, there is no doubt that an effective interaction of the type (8.25) is generated. but As it respects $SU_L(N_f) \times SU_L(N_f)$ but breaks $U_A(1)$ symmetry, it consitutes the solution to the $U(1)$ problem.

A more quantitative analysis of the solution requires a certain analytic consideration based on $1/N_c$ expansion followed by a numerical analysis based on the lattice QCD.

8.1. Determination of gaugino condensates in supersymmetric Yang-Mills theories: "$\frac{5}{4}$ puzzle"

In $\mathcal{N}=1$ supersymmetric Yang-Mills (SYM) theories, with fields (A^a_μ, λ^a) it is possible to calculate the value of gluino condensates like

$$\langle T(\lambda\lambda(x_1)\dots\lambda\lambda(x_{N_f})\rangle \tag{8.26}$$

which is constant and position independent because of SUSY[3]. Using the cluster decomposition principle, we can write

$$\langle T(\lambda\lambda(x_1)\dots\lambda\lambda(x_{N_f})\rangle = \langle\lambda\lambda\rangle^N = const\,\Lambda^{3N} = const\,e^{-\frac{8\pi^2}{g_0^2(M)}}M^{3N} \tag{8.27}$$

with $const \neq 1$ (strong coupling instanton). So we have N vacua characterized by

$$\langle\lambda\lambda\rangle = const\,e^{\frac{2\pi ik}{N}}\Lambda^3 \tag{8.28}$$

with $U_A(1) \supset \mathbb{Z}_2N \xrightarrow{\langle\lambda\lambda\rangle} \mathbb{Z}_2$ the pattern of symmetry breaking.

It turns out that the "vacuum disentanglement" argument used in [] does not give the correct value of the gaugino condensate. In the case of $SU(2)$ SYM, there is a famous

the calculation of $\langle\lambda\lambda\rangle$ along flat directions for $\langle Q\rangle \gg \lambda$, and the limit $m_Q \to \infty$ and $\Lambda \to \infty$ with Λ_{YM} fixed. The result is, for the $SU(N)$ Yang-Mills theory

$$\langle\lambda\lambda\rangle = e^{\frac{2\pi ik}{N}}\Lambda^3, \tag{8.29}$$

[3] In fact $\lambda\lambda$ is the lowest component of $-W^\alpha W_\alpha$ which is a chiral superfield, so $[\bar{Q}, \lambda\lambda] = 0$. This implies, if the vacuum is invariant under supersymmetry, $\frac{\partial}{\partial x_1}\langle\lambda\lambda(x_1)\lambda\lambda(x_2)\dots\rangle = \langle[P, \lambda\lambda(x_1)]\lambda\lambda(x_2)\dots\rangle = \langle[\{Q_\alpha, \bar{Q}_{\dot\alpha}\}, \lambda\lambda(x_1)]\lambda\lambda(x_2)\dots\rangle = \langle\lambda\lambda(x_1)Q_\alpha\bar{Q}_{\dot\alpha}\lambda\lambda(x_2)\dots\rangle = \langle\lambda\lambda(x_1)Q_\alpha\lambda\lambda(x_2)\dots\bar{Q}_{\dot\alpha}\rangle = 0$.

while for other groups

$$
\begin{aligned}
\left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{SU(r+1)} &= \Lambda^3_{\mathcal{N}=1} \, , \\
\left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{SO(2r+1)} &= 2^{\frac{4}{2r-1}-1} \Lambda^3_{\mathcal{N}=1} \, , \\
\left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{USp(2r)} &= 2^{1-\frac{2}{r+1}} \Lambda^3_{\mathcal{N}=1} \, , \\
\left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{SO(2r)} &= 2^{\frac{2}{r-1}-1} \Lambda^3_{\mathcal{N}=1} \, ,
\end{aligned}
\tag{8.30}
$$

up to the phase factor $e^{2\pi i\, k/T_G}$ that distinguishes the T_G vacua, in agreement with [**?**, **?**, **?**].

9. Homotopy Groups

9.1. Differential manifolds

Def.

An n-dimensional differential manifold is a set M of the points with the following properties:

(i) The set M is the union of a finite or numerable number of neighborhoods U_q;

(ii) Each of U_q is endowed with coordinates x^α_q, $\alpha = 1, \ldots, n$, called local coordinates;

(iii) If $U_q \cup U_p \neq \emptyset$, then $U_q \cup U_p$ itself forms a coordinate neighborhood,

10. Elliptic Functions

11. Monopoles, Dyons and Vortices in Non-Abelian Gauge Theories

In this section the issues concerning monopoles and vortices in spontaneopusly broken gauge theoris are reviewed. In particular it is shown that the concept of nonabelian monopoles and nonabelian vortices is an intrinsically quantum mechanical one, and requires an appropriate massless flavors for their very existence.

11.1. Semiclassical Results

We study the general setting of a spontaneously broken gauge theory, with its gauge group G broken as

$$G \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} H \tag{11.1}$$

by some scalar vevs, where H is in general non-Abelian. The aim of this section is to identify the relevant homotopy group elements and prepare for the subsequent sections where we explicitly construct the semi-classical BPS monopole solutions and compare them with the infrared degrees of freedom appearing in the softly broken $\mathcal{N}=2$ gauge theories.

In order to have a nontrivial finite-energy configuration, the scalar fields and gauge field must behave asymptotically as

$$\mathcal{D}\phi \stackrel{r\to\infty}{\longrightarrow} 0 \quad \Rightarrow \quad \phi \sim U\cdot\langle\phi\rangle\cdot U^{-1}, \qquad A_i^a \sim U\cdot\partial_i U^{\dagger} \to \epsilon_{aij}\frac{r_j}{r^3}G(r), \tag{11.2}$$

representing nontrivial elements of $\Pi_2(G/H)$. By an appropriate choice of gauge, the function $G(r)$ can be taken as

$$G(r) = \beta_i T_i, \qquad T_i \in \text{Cartan Subalgebra of } H. \tag{11.3}$$

Topological quantization leads to the result [11] that the "charges" β_i take values such that

$$\exp 4\pi\, i \sum_1^r \beta_i T_i = 1, \tag{11.4}$$

where r is the rank of H. By commuting this relation with the nondiagonal generators E_α and by using

$$[\, T_i, E_\alpha\,] = \alpha_i\, E_\alpha, \tag{11.5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_r)$ are the root vectors of H, one finds that

$$2\,\beta\cdot\alpha = \mathbb{Z}. \tag{11.6}$$

This relation shows that

$SU(N)/Z_N$	$\Leftrightarrow$	$SU(N)$
$SO(2N)$	$\Leftrightarrow$	$SO(2N)$
$SO(2N+1)$	$\Leftrightarrow$	$USp(2N)$

Table 2: Some examples of dual pairs of groups

which are weight vectors of the group $\tilde{H}$ where $\tilde{H}$ = dual of H. The dual of a group (whose roots vectors are α's) is defined by the root vectors which span the dual lattice, i.e., $\tilde{\alpha} = \alpha/\alpha^2$. Examples of pairs of the duals are:

Here we consider the case in which $\mathbf{h}$ is orthogonal to the root vectors of a $SU(r)$ subgroup. The simplest way to detect the presence of the non-Abelian monopoles is to consider various $SU(2)$ subgroups generated by

$$t_1 = \frac{1}{\sqrt{2\alpha^2}}(E_\alpha + E_{-\alpha}); \qquad t_2 = -\frac{i}{\sqrt{2\alpha^2}}(E_\alpha - E_{-\alpha}); \qquad t_3 = \alpha^* \cdot \mathbf{H}, \quad (11.7)$$

where α is a root vector associated with broken generators $E_{\pm\alpha}$ and $\alpha^* \equiv \alpha/\alpha^2$. In particular we consider those $SU(2)$ groups which do not commute with the unbroken subgroup $SU(r)$. In the notation of Eq.(11.12) these correspond to $SU(2)$ subgroups acting in the $[i-k]$ subspaces, where $i = 1, 2, \dots, r,$ and $k = r+1, r+2, \dots n_c$. The symmetry breaking (11.11) induces the Higgs mechanism in such an $SU(2)$ subgroup,

$$SU(2) \Longrightarrow U(1). \qquad (11.8)$$

By embedding the known 't Hooft-Polyakov monopole [**?**] lying in this subgroup, and adding a constant term for ϕ so that it behaves correctly asymptotically, one easily constructs a solution of the $SU(n_c)$ equation of motion (see E. Weinberg [**?**]):

$$A_i(\mathbf{r}) = A_i^a(\mathbf{r}, \mathbf{h}\cdot\alpha)\, t_a; \qquad \phi(\mathbf{r}) = \chi^a(\mathbf{r}, \mathbf{h}\cdot\alpha)\, t_a + (\mathbf{h} - (\mathbf{h}\cdot\alpha)\alpha^*) \cdot \mathbf{H}, \quad (11.9)$$

where

$$A_i^a(\mathbf{r}) = \epsilon_{aij} \frac{r^j}{r^2} A(r); \qquad \chi^a(\mathbf{r}) = \frac{r^a}{r}\chi(r), \qquad \chi(\infty) = \mathbf{h}\cdot\alpha \qquad (11.10)$$

is the standard 't Hooft-Polyakov solution. Note that $\phi(\mathbf{r} = (0, 0, \infty)) = \phi_0$.

To be concrete we consider a general (supersymmetric or non supersymmetric) $SU(n_c)$ gauge theory with an appropriate set of scalar fields in the adjoint representation. As will be mentioned at the end, our analysis applies equally well to other gauge groups. We assume that the minimum of the potential is such that the gauge group is broken spontaneously as

$$SU(n_c) \to SU(r) \times U(1)^{n_c - r}. \tag{11.11}$$

For instance the VEV of a scalar can be taken in the diagonal form

$$\langle \phi \rangle = \begin{pmatrix} v_0 \mathbf{1}_{r\times r} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & v_{r+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & \dots & v_{n_c} \end{pmatrix}, \qquad r\, v_0 + \sum_{j=r+1}^{n_c} v_j = 0, \tag{11.12}$$

where v_i's are all different. Let us write the asymptotic Higgs field more compactly as

$$\phi_0 = \mathbf{h} \cdot \mathbf{H}, \tag{11.13}$$

where the $n_c - 1$ rank vector $\mathbf{h}$ describes the scalar VEV, while $\mathbf{H}$ represents the generators in the Cartan subalgebra of $SU(n_c)$. If $\mathbf{h}$ had non-zero inner products with all of the root vectors of $SU(n_c)$ then the gauge group would be maximally broken to $U(1)^{n_c-1}$ group and Abelian monopoles having respective $U(1)$ charges would appear. We have nothing to add about such a system.

Here we consider the case in which $\mathbf{h}$ is orthogonal to the root vectors of a $SU(r)$ subgroup. The simplest way to detect the presence of the non-Abelian monopoles is to consider various $SU(2)$ subgroups generated by

$$t_1 = \frac{1}{\sqrt{2\alpha^2}}(E_\alpha + E_{-\alpha}); \qquad t_2 = -\frac{i}{\sqrt{2\alpha^2}}(E_\alpha - E_{-\alpha}); \qquad t_3 = \alpha^* \cdot \mathbf{H}, \tag{11.14}$$

where α is a root vector associated with broken generators $E_{\pm\alpha}$ and $\alpha^* \equiv \alpha/\alpha^2$. In particular we consider those $SU(2)$ groups which do not commute with the unbroken subgroup $SU(r)$. In the notation of Eq.(11.12) these correspond to $SU(2)$ subgroups acting in the $[i-k]$ subspaces, where $i = 1, 2, \dots, r,$ and $k = r+1, r+2, \dots n_c$. The symmetry breaking (11.11) induces the Higgs mechanism in such an $SU(2)$ subgroup,

$$SU(2) \Longrightarrow U(1). \tag{11.15}$$

By embedding the known 't Hooft-Polyakov monopole [**?**] lying in this subgroup, and adding a constant term for ϕ so that it behaves correctly asymptotically, one easily constructs a solution of the $SU(n_c)$ equation of motion (see E. Weinberg [?]):

$$A_i(\mathbf{r}) = A_i^a(\mathbf{r}, \mathbf{h}\cdot\alpha)\, t_a; \qquad \phi(\mathbf{r}) = \chi^a(\mathbf{r}, \mathbf{h}\cdot\alpha)\, t_a + (\mathbf{h} - (\mathbf{h}\cdot\alpha)\alpha^*)\cdot\mathbf{H}, \quad (11.16)$$

where

$$A_i^a(\mathbf{r}) = \epsilon_{aij}\frac{r^j}{r^2}A(r); \qquad \chi^a(\mathbf{r}) = \frac{r^a}{r}\chi(r), \qquad \chi(\infty) = \mathbf{h}\cdot\alpha \qquad (11.17)$$

is the standard 't Hooft-Polyakov solution. Note that $\phi(\mathbf{r} = (0,0,\infty)) = \phi_0$.

The mass of this monopole for the minimum magnetic charge is given by the standard formula, in the case of BPS monopoles,

$$M = \frac{4\pi}{g}\mathbf{h}\cdot\alpha = \frac{4\pi}{g}|v_0 - v_k|. \qquad (11.18)$$

By an appropriate field redefinition v_0 can be always taken to be positive. Also, for generic, unequal values of v_i, it is possible, by using a Weyl transformation, to take the scalar VEV so that

$$|v_0 - v_{r+1}| < |v_0 - v_k|, \qquad k = r+2, r+3, \ldots, n_c. \qquad (11.19)$$

By considering various $SU(2)$ subgroups acting on $[i, r+1]$ subspaces, where $i = 1, 2, \ldots, r$, we find that *there are precisely r degenerate solutions with the same mimimum mass,*

$$M = \frac{4\pi}{g}|v_0 - v_{r+1}|. \qquad (11.20)$$

They are transformed to each other by the Weyl transformations. By construction these solutions carry also a unit (magnetic) charge with respect to the $U_0(1)$ gauge group, which is generated by

$$Q_0 = \begin{pmatrix} \frac{1}{r}\mathbf{1} & 0 & \ldots & \ldots \\ 0 & -1 & 0 & \ldots \\ \vdots & 0 & 0 & \ldots \\ \vdots & 0 & \ldots & \ddots \end{pmatrix} \qquad (11.21)$$

The system, furthermore, has $n_c - r - 1$ Abelian monopoles, each with the minimal charge in

$$Diag\, Q_\ell = [\,\underbrace{0, 0, \ldots, 0}_{\ell}, 1, -1, 0, \ldots, 0\,], \qquad r \le \ell \le n_c - 1, \qquad (11.22)$$

and with mass

$$M_\ell = \frac{4\pi}{g}|v_\ell - v_{\ell+1}|. \tag{11.23}$$

For appropriate choice of the scalar vacuum expectation values (VEVS) (and arranging them appropriately by Weyl transformations) there are thus an r-plet of "non-Abelian" monopoles and $n_c - r - 1$ Abelian monopoles with minimum charges and minimum masses that are stable .

12. Seiberg-Witten Solution of $\mathcal{N} = 2$ Gauge Theories

13. Quantum Behavior of Solitons in $\mathcal{N} = 2$ Gauge Theories

14. Solutions for the Chiral Condensates for $\mathcal{N} = 1$ Gauge Theories

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