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Advent of non-Abelian Vortices

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Plan

I. Non-Abelian vortices in 3+1 dim gauge theories

- Topology and duality in non-Abelian gauge theories
- Supersymmetry
- Vortex solutions with non-Abelian moduli
  SU(2)xU(1) models with $N_f = 2$ flavors:
  U(N) vortices, higher-winding vortices, non-BPS, etc.
- Vortex-monopole connection
  (homotopy sequence and symmetry)

II. Non-Abelian vortices: generalizations

- Vortices in general gauge systems
- Vortices with product moduli space
- Fractional vortices
- Monopole-vortex complex

Lecture 1

Electromagnetic duality and topological solitons

- Vacuum Maxwell equations

$$\nabla\bullet(E+ i\, B) =0; \;\; \nabla \times (E+ i\, B)= i\, \partial_t\, (E+ i\, B)$$

inv under $E+ i\, B \rightarrow e^{i\,\phi}\, (E+ i\, B)$ (broken by charges)

- Magnetic monopole possible (Dirac 1931) in quantum field theory if

$g \,.\, g_m = n/2, \quad n=0,1,2,...$ quantization of electric charges

't Hooft, Polyakov

- Soliton monopoles in spont. broken gauge theories (1974)

GUT (grand-unified models) monopoles?

- Soliton vortices (Abrikosov '57 , Nielsen-Olesen '74)

(superconductor, Landau-Ginzburg model, Abelian Higgs model)

- Other applications in condensed matter physics / cosmology, etc

- Confinement ~ dual superconductor ? Nambu, 't Hooft, Mandelstam '80

Quark Confinement in QCD = Dual superconductor

.... of Non-Abelian variety ?

$G \rightarrow H$ $\Pi_2(G/H) \neq 1$

? How to generalize to H non-Abelian

{ 't Hooft-Polyakov monopole ('74)

ANO vortex ('73) $H=U(1)$

$\Pi_1(H) \neq 1$

Key developments:

- Quantum behavior of Abelian and non-Abelian monopoles

('94 -'05) Seiberg-Witten, Argyres,Douglas, Shenker Carlino,Konishi,Murayama

- Discovery of non-Abelian vortices ('03-)

Hanany-Tong, Auzzi,Bolognesi,Evslin,Konishi,Yung

⇨ Rich variety of new results

Konishi, a review **hep-th/0702102**

('04-'10) Pisa, Tokyo, Minnesota, Cambridge ...

Non-Abelian Vortices

Abelian Higgs model and ANO vortex

Abrikosov '56
Nielsen-Olesen '73

$L= -(1/4\, g^2)\,(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V,$

$V= \lambda\, (\, |\phi|^2 - v^2\,)^2\ /2$

$D_\mu = \partial_\mu - i\, A_\mu$

$D\phi \rightarrow 0; \quad |\phi|^2 \rightarrow v^2$

$\phi \sim v\, e^{i\,\phi}$ far from the vortex core

$\Pi_1(U(1))=Z$

"ANO" vortex

- $\lambda > g^2/2$ type I
- $\lambda < g^2/2$ type II
- $\lambda = g^2/2$ BPS *

* BPS-saturated
(Bogomolnyi-Prasad-Sommerfield)
= Self dual case

Extended Abelian Higgs (EAH) model

Vachaspati, Achucarro, ...

$|\phi|^2 \Rightarrow \Sigma_i\, |\phi_i|^2$

$\Pi_1(CP^{N-1})= 1$

but $\Pi_2(CP^{N-1})= Z$

- $\lambda > g^2/2$ type I: ANO stable
- $\lambda < g^2/2$ type II: ANO unstable
- $\lambda = g^2/2$ BPS: semi-local vortices

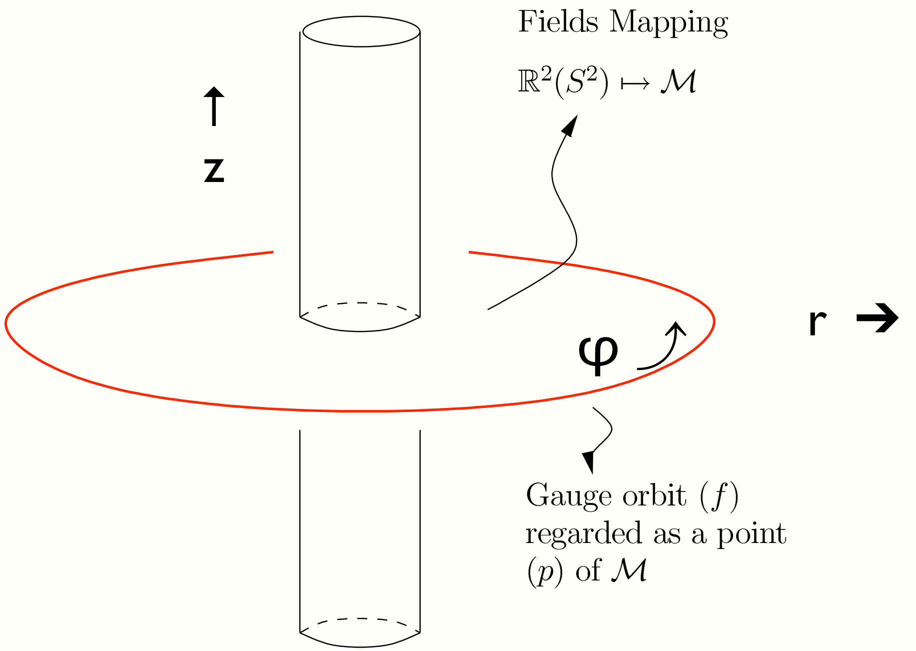


Fig. 2:

Cylindrical coordinates
r, φ, z

M= vacuum configurations {ϕ}; F= gauge orbits

f, p = point of F, $\mathscr{M}$, respectively

$\mathscr{M}$ = vacuum moduli space = M/F

$\mathscr{M}$ = S/S =1, AH

= S^{2N-1} /S = CP^{N-1}, EAH

- •A vortex defined at each point p of the base space $\mathscr{M}$ (vacuum degeneracy)
- Vortex solutions possess in general nontrivial vortex moduli $\mathscr{V}$

  A symmetry broken by the individual soln (e.g. $\mathbf{R}^2$ for AH); or due to $\mathscr{M}$
- Semilocal Vortex ~ sigma model lump ($\Pi_2(\mathscr{M})$)

Non-Abelian vortex *

Hanany-Tong, '03
Auzzi-Bolognesi-Evslin-Konishi-Yung. '03

- $H \overset{\Phi_2 \neq 0}{\Rightarrow} \mathbf{1}$ with $\pi_1(H) \neq 1$

H: non-Abelian (**)

Shifman-Yung, ... (Minnesota).
Eto-Nitta-Ohashi-Sakai- ... (TiTech, Tokyo).
Tong, (Cambridge).
Pisa group, '03-'09

- ** not sufficient.

N.B. $H = SU(N)/Z_N \Rightarrow Z_N$ vortex ! $(\pi_1(H) = Z_N)$

- Need a global (flavor) symmetry:
  U(N) theory with $N_f = N$ squarks in the fundamental repres. of SU(N)

- Color-flavor locked vacuum

$$\langle q \rangle \propto 1_{N\times N}$$

$$(q)^i_\alpha = \begin{pmatrix} q_1^{(1)} & q_1^{(2)} & \cdots & & q_1^{(N)} \\ q_2^{(1)} & q_2^{(2)} & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & & \vdots \\ q_N^{(1)} & q_N^{(2)} & \cdots & & q_N^{(N)} \end{pmatrix}$$

* Vortex solutions with continuous non-Abelian moduli

U(N) model (with N$_f$ =N "flavors" of complex scalar fields -- squarks)

$$\mathcal{L} \;=\; \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \frac{2}{g^2}\mathcal{D}_\mu\,\phi^\dagger\,\mathcal{D}^\mu\phi - \mathcal{D}_\mu\,H\,\mathcal{D}^\mu H^\dagger - \lambda\left(c\,\mathbf{1}_N - H\,H^\dagger\right)^2\right]$$
$$+\;\;\mathrm{Tr}\;[\,(H^\dagger\phi - M\,H^\dagger)(\phi\,H - H\,M)\,]$$

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\nu + i\,[W_\mu, W_\nu]\text{ and }\mathcal{D}_\mu H = (\partial_\mu + i\,W_\mu)\;H,$$

$(H)^i_\alpha \equiv q^i_\alpha$: N complex scalar fields in the fundamental representation of SU(N), written in color-flavor mixed matrix form

ϕ A complex scalar field in the adjoint representation of SU(N)

$M = diag\,(m_1, m_2, \ldots, m_N)$ is the mass matrix for the squarks q

- For a critical coupling constant $\lambda = \dfrac{g^2}{4}$ *) BPS (self-dual) (automatic in Susy)

the model can be regarded as a truncation of the bosonic sector of a N=2 supersymmetric model, with $(H)^i_\alpha \equiv q^i_\alpha$, $\tilde{q}^\alpha_i \equiv 0$

- In this case c comes from the Fayet-Iliopoulos term L = c V|$_D$

- For unequal masses

$$\langle\phi\rangle = M = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & m_N \end{pmatrix}$$

breaks U(N) ➔ U(1)^N

U(1), s broken by the squark vac. exp. value ➔ ANO vortex nothing really new

Equal mass case: Non-Abelian Vortices:

Auzzi-Bolognesi-Evslin-Konishi-Yung,
Hanany-Tong, Shifman-Yung, Eto, et. al.

$$\langle \phi \rangle = m\, \mathbf{1}_N, \qquad \langle H \rangle = \sqrt{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Color-flavor locked phase

- The SU(N)xU(1) gauge group broken completely;
- The $SU(N)_{C+F}$ flavor symmetry intact   U ⟨H⟩ U^{-1} =   ⟨H⟩

color   flavor

- The BPS (self-dual) vortex equations

z = x + i y

$\partial_z = (\partial_x - i\, \partial_y)/2$

$$(\mathcal{D}_1 + i\mathcal{D}_2)\ H = 0, \quad F_{12} + \frac{g^2}{2}\left(c\, \mathbf{1}_N - H\, H^\dagger\right) = 0.$$

- The solutions

Eto-Nitta-Ohashi-Sakai...

holomorphic

$$H = S^{-1}(z,\bar z)\, H_0(z), \quad W_1 + i\, W_2 = -2\, i\, S^{-1}(z,\bar z)\, \bar\partial_z S(z,\bar z).$$

S: complex extension of U(N)~ GL(N,C)

- Ω = S $S^\dagger$ satisfies the master equation

$$\partial_z\, (\Omega^{-1}\partial_{\bar z}\, \Omega) = \frac{g^2}{4}\, (c\, \mathbf{1}_N - \Omega^{-1}\, H_0\, H_0^\dagger).$$

any non-singular holomorphic
NxN matrix

- The moduli matrix H_0 defined up to V equivalence relations

$$H_0(z) \to V(z)\, H_0(z), \qquad S(z,\bar z) \to V(z)\, S(z,\bar z),$$

The problem : Master (gauge field) equation

$\Omega = S\, S^\dagger$ S: complex extension of U(N)~ GL(N,C): any regular NxN matrix **

(i) Solve for the Hermitian NxN matrix Ω, $\Omega^\dagger = \Omega$

$$\partial_z \left(\Omega^{-1} \partial_{\bar z}\, \Omega\right) = \frac{g^2}{4} \left(c\, 1_N - \Omega^{-1}\, H_0\, H_0^\dagger\right).$$

(g, c are constants, set to 1), given a holomorphic moduli matrix $H_0(z)$, with the boundary condition

$$\Omega \rightarrow (1/c)\ H_0\, H_0^\dagger\ , \qquad |z| \rightarrow \infty$$

det $H_0(z) \sim z^k + \ldots$

k= the winding number

(ii) Show the existence and uniqueness of the solution for each H_0

e.g.,
$$H_0^{(1,0,\ldots,0)} = \begin{pmatrix} z - z_0 & 0 & 0 & \ldots & 0 \\ b_1 & 1 & 0 & \ldots & 0 \\ b_2 & 0 & \ddots & & 0 \\ \vdots & 0 & \ldots & & 0 \\ b_{N-1} & 0 & 0 & \ldots & 1 \end{pmatrix}$$

(z_0, b_i are complex moduli parameters)

(**) for other gauge groups see later

U(2) ~ SU(2)xU(1) model as the low-energy effective theory from SU(3) theory

Auzzi-Bolognesi-Evslin-Konishi-Yung,

Adjoint scalar VEV

$$\phi = -\frac{1}{\sqrt{2}}\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -2m \end{pmatrix}$$

SU(3) ➔ SU(2)xU(1)/Z_2

$$S = \int d^4x \left[\frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2}\left(F^8_{\mu\nu}\right)^2 + \left|\nabla_\mu q^A\right|^2 \right.$$

$$\left. + \frac{g_2^2}{8}\left(\bar{q}_A \tau^a q^A\right)^2 + \frac{g_1^2}{24}\left(\bar{q}_A q^A - 2\xi\right)^2 \right],$$

Bogomolnyi completion
for static vortex soln

$$T = \int d^2x \left(\sum_{a=1}^{3} \left[\frac{1}{2g_2} F_{ij}^{(a)} \pm \frac{g_2}{4}\left(\bar{q}_A \tau^a q^A\right)\epsilon_{ij}\right]^2 \right.$$

$$\left. + \left[\frac{1}{2g_1}F_{ij}^{(8)} \pm \frac{g_1}{4\sqrt{3}}\left(|q^A|^2 - 2\xi\right)\epsilon_{ij}\right]^2 + \frac{1}{2}\left|\nabla_i\, q^A \pm i\epsilon_{ij}\nabla_j\, q^A\right|^2 \pm \frac{\xi}{\sqrt{3}}\tilde{F}^{(8)} \right)$$

$$\tilde{F}^{(8)} \equiv \tfrac{1}{2}\epsilon_{ij}F_{ij}^{(8)}$$

Non-Abelian BPS
(self-dual) equations

$$\frac{1}{2g_2}F_{ij}^{(a)} + \frac{g_2}{4}\varepsilon\left(\bar{q}_A \tau^a q^A\right)\epsilon_{ij} = 0, \qquad a = 1, 2, 3;$$

$$\frac{1}{2g_1}F_{ij}^{(8)} + \frac{g_1}{4\sqrt{3}}\varepsilon\left(|q^A|^2 - 2\xi\right)\epsilon_{ij} = 0; \qquad i, j = 1, 2$$

$$\nabla_i\, q^A + i\varepsilon\epsilon_{ij}\nabla_j\, q^A = 0, \qquad A = 1, 2, \ldots, N_f.$$

Vortex Ansatz and profile fns (SU(2)xU(1) case)

($A^8 = A^0$)

q(r, φ)= $\begin{pmatrix} e^{i\,n\,\varphi}\phi_1(r) & 0 \\ 0 & e^{i\,k\,\varphi}\phi_2(r) \end{pmatrix}$

$$A_i^3(x) = -\varepsilon\epsilon_{ij}\,\frac{x_j}{r^2}\,\left((n-k) - f_3(r)\right),$$

$$A_i^8(x) = -\sqrt{3}\;\varepsilon\epsilon_{ij}\,\frac{x_j}{r^2}\,\left((n+k) - f_8(r)\right)$$

Self-dual equations:

$$r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_1(r) - \frac{1}{2}\left(f_8(r) + f_3(r)\right)\phi_1(r) = 0,$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_2(r) - \frac{1}{2}\left(f_8(r) - f_3(r)\right)\phi_2(r) = 0,$$

$$-\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}f_8(r) + \frac{g_1^2}{6}\,\left(\phi_1(r)^2 + \phi_2(r)^2 - 2\xi\right) = 0,$$

$$-\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}f_3(r) + \frac{g_2^2}{2}\,\left(\phi_1(r)^2 - \phi_2(r)^2\right) = 0.$$

Boundary conditions:

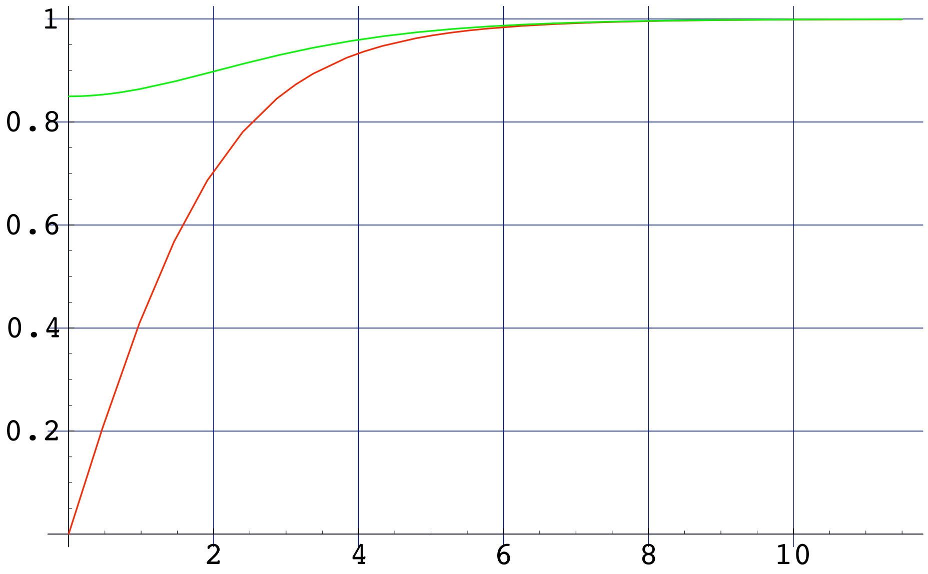
$$f_3(0) = \varepsilon_{n,k}\,(n-k)\,, \quad f_8(0) = \varepsilon_{n,k}\,(n+k)\,,$$

$$f_3(\infty) = 0, \quad f_8(\infty) = 0$$

$$\phi_1(\infty) = \sqrt{\xi}, \quad \phi_2(\infty) = \sqrt{\xi}$$

ϕ regular everywhere (e.g., $\phi_1(0) = 0$, if $n \neq 0,\;\; k = 0$)

A minimal vortex: $n=1$; $k=0$



$\phi_1(r)$, $\phi_2(r)$

$f_3(r)$, $f_8(r)$,

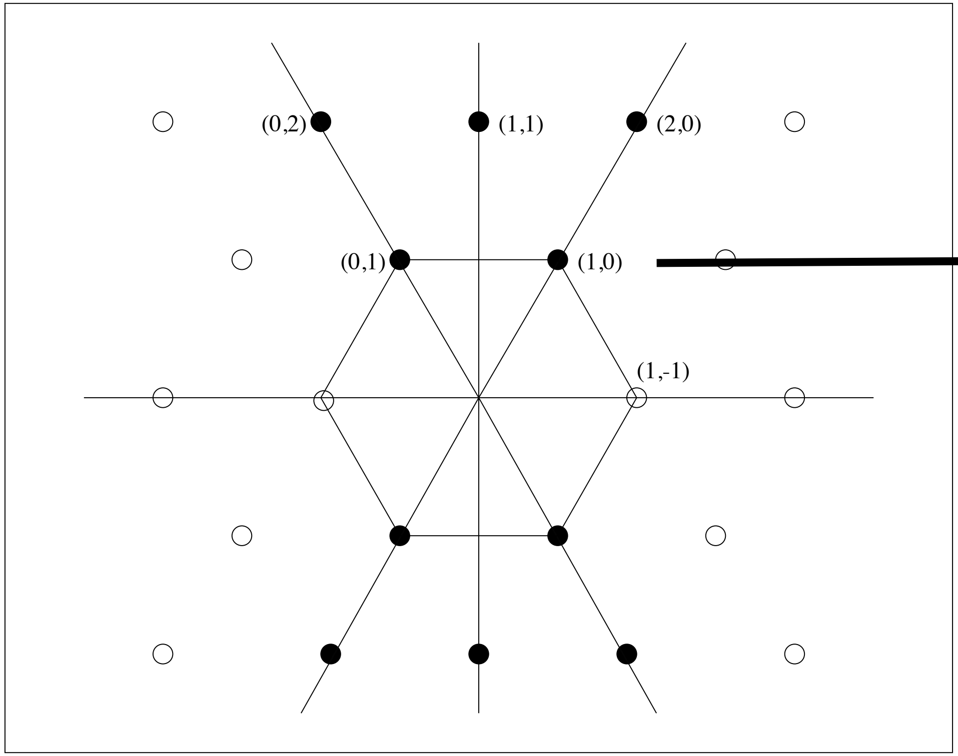
Vortex tension and degeneracy

$$T = \int d^2x \left(\sum_{a=1}^{3} \left[\frac{1}{2g_2} F_{ij}^{(a)} \pm \frac{g_2}{4}\left(\bar{q}_A \tau^a q^A\right) \epsilon_{ij} \right]^2 \right.$$

$$\left. + \left[\frac{1}{2g_1} F_{ij}^{(8)} \pm \frac{g_1}{4\sqrt{3}} \left(|q^A|^2 - 2\xi\right) \epsilon_{ij} \right]^2 + \frac{1}{2} \left| \nabla_i \, q^A \pm i\epsilon_{ij} \nabla_j \, q^A \right|^2 \pm \frac{\xi}{\sqrt{3}} \tilde{F}^{(8)} \right)$$

where $\tilde{F}^{(8)} \equiv \frac{1}{2}\epsilon_{ij} F_{ij}^{(8)}$ ⇨ Tension:

$$T_{n,k} = 2\pi \, \xi \, |n + k|.$$



e.g. , (1,0) and (0,1) vortices have the same tension

Actually, the vortex degeneracy is actually larger

$$S = s\, S', \quad \omega = s s^\dagger$$

$$T = 2\xi \int d^2x \, \partial \bar{\partial} \, log \, \omega$$

Orientational zero modes

Exact SU(2)$_{C+F}$ symmetry of the system (eq. of motion and the vacuum) broken by individual vortex solution:

SU(2)$_{C+F}$ ➔ U(1)

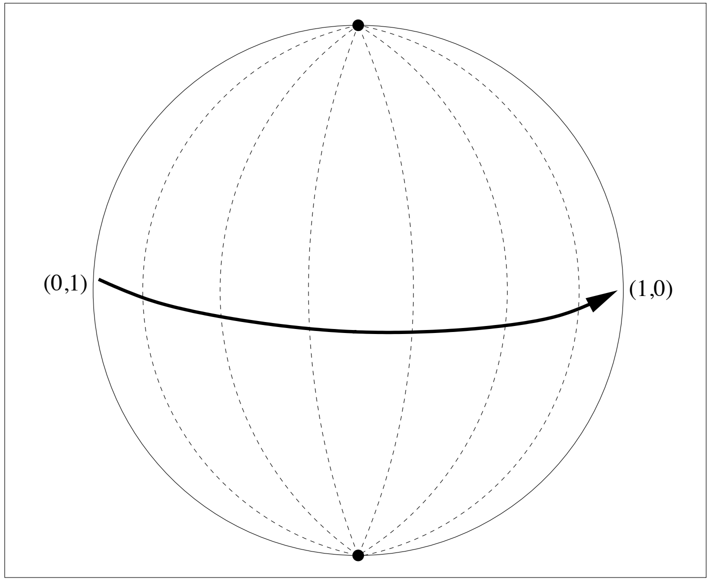
Orientational zeromodes $U \subset SU(2)/U(1) \sim \mathrm{CP}^1 \sim S^2$

$$q^{kA} = U \begin{pmatrix} e^{i\varphi}\phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = e^{\frac{i}{2}\varphi(1+n^a\tau^a)} U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1}$$

$$\mathrm{A}_i(x) = U[-\frac{\tau^3}{2}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_3(r)]]U^{-1} = -\frac{1}{2}\,n^a\tau^a\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_3(r)],$$

(Tension invariant)

$$A_i^8(x) = -\sqrt{3}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_8(r)],$$



$$U\tau^3U^{-1} = n^a\tau^a, \quad A_\mu = A_\mu^a\tau^a/2.$$

$$n^2 = 1,$$

parametrizes S^2

Moduli-matrix formalism

Vortex of minimum winding (k=1):

2x2 matrix holomorphic in z= x+i y

$$H = S^{-1}\, H_0(z), \qquad A_1 + i A_2 = -2\, i\, S^{-1}(z,\bar{z})\bar{\partial}_z S(z,\bar{z})$$

det $H_0 \sim$ z

vortex center

vortex orientation

⇨ $$\boldsymbol{H}_0^{(1,0)}(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{z}-\boldsymbol{z}_0 & \mathbf{0} \\ -\boldsymbol{b}' & \mathbf{1} \end{pmatrix}, \qquad \boldsymbol{H}_0^{(0,1)}(\boldsymbol{z}) = \begin{pmatrix} \mathbf{1} & -\boldsymbol{b} \\ \mathbf{0} & \boldsymbol{z}-\boldsymbol{z}_0 \end{pmatrix}$$

change of the local patches by $H^{(1,0)} \rightarrow H^{(0,1)} = V\ H^{(1,0)}$

$$V = \begin{pmatrix} 0 & -1/b' \\ b' & z-z_0 \end{pmatrix} \in GL(2,\mathbf{C}).$$ except at b' =0

$$\boldsymbol{b} = \frac{\mathbf{1}}{\boldsymbol{b}'}$$ the inhomogeneous coordinate of the Riemann sphere S^2 = CP^1

- In general, the vortex moduli space is a complex manifold. V transformations provide the transition functions among the local coordinates

- $U(2)$ with $N_f = 2$ ($a_0 = 1/b_0$; $CP^1 \sim SU(2)/U(1)$), $H_0(z) \sim V(z)\, H_0(z)$

$$H_0^{(1,0)} \simeq \left(\begin{array}{cc} z - z_0 & 0 \\ -b_0 & 1 \end{array} \right); \qquad H_0^{(0,1)} \simeq \left(\begin{array}{cc} 1 & -a_0 \\ 0 & z - z_0 \end{array} \right),$$

- ($U(2)$ with $N_f = 2$)

$$H_0 \to U\, H_0\, U^{-1} \sim H_0', \qquad U = \left(\begin{array}{cc} \alpha & \beta \\ -\beta^* & \alpha^* \end{array} \right)$$

$$a_0 \to \frac{\alpha a_0 + \beta}{\alpha^* + \beta^* a_0}.$$

- OK with

$$\left(\begin{array}{c} a_1 \\ a_2 \end{array} \right) \to \left(\begin{array}{cc} \alpha & \beta \\ -\beta^* & \alpha^* \end{array} \right) \left(\begin{array}{c} a_1 \\ a_2 \end{array} \right), \quad \frac{a_1}{a_2} = a_0.$$

- Precisely the $SU(2)$ transformation of a two-state quantum mechanical system

$$a_1\, |1\rangle + a_2\, |2\rangle.$$

Vortices in U(N) = SU(N)xU(1)/Z$_N$ theory

U(3)

$$q^{kA} = \begin{pmatrix} e^{i\,n\,\varphi}\phi_1(r) & 0 & 0 \\ 0 & e^{i\,k\,\varphi}\phi_2(r) & 0 \\ 0 & 0 & e^{i\,p\,\varphi}\phi_3(r) \end{pmatrix}, \qquad \phi_1, \phi_2, \phi_3 \to \sqrt{\xi}, \qquad r \to \infty.$$

$$A_i^3(x) = -\epsilon_{ij}\,\frac{x_j}{r^2}\Big((n-k) - f_3(r)\Big),$$

$$A_i^8(x) = -\frac{1}{\sqrt{3}}\epsilon_{ij}\,\frac{x_j}{r^2}\Big((n+k-2p) - f_8(r)\Big), \qquad i, j = 1, 2$$

$$A_i(x) = -\frac{1}{3}\epsilon_{ij}\,\frac{x_j}{r^2}\Big((n+k+p) - f_0(r)\Big).$$

Self-dual equations
in terms of the profile
functions

$$r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_1(r) - \Big(\frac{1}{2}f_3(r) + \frac{1}{6}f_8(r) + \frac{1}{3}f_0(r)\Big)\phi_1(r) = 0,$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_2(r) - \Big(-\frac{1}{2}f_3(r) + \frac{1}{6}f_8(r) + \frac{1}{3}f_0(r)\Big)\phi_2(r) = 0,$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_3(r) - \Big(-\frac{1}{3}f_8(r) + \frac{1}{3}f_0(r)\Big)\phi_3(r) = 0,$$

$$-\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}f_3(r) + g^2\left(\frac{1}{2}\phi_1(r)^2 - \frac{1}{2}\phi_2(r)^2\right) = 0.$$

$$-\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}f_8(r) + g^2\left(\frac{1}{2}\phi_1(r)^2 + \frac{1}{2}\phi_2(r)^2 - \phi_3(r)^2\right) = 0,$$

$$-\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}f_0(r) + 3e^2\left(\phi_1(r)^2 + \phi_2(r)^2 + \phi_3(r)^2 - 3\,\xi\right) = 0.$$

the tension is given by

$$T_{n,k,p} = 2\,\pi\,\xi\,|\,n+k+p\,|.$$

But for k=1 e.g., (1,0,0) vortex in U(3) theory,

can set $\phi_2 = \phi_3 = \phi, \qquad f_3 = f_8 = f_{NA}$

the vortex equations simplify to

$$
\begin{aligned}
r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_1(r) - \Big(\frac{2}{3}f_{NA}(r) + \frac{1}{3}f(r)\Big)\phi_1(r) &= 0,\\
r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi(r) - \Big(-\frac{1}{3}f_{NA}(r) + \frac{1}{3}f(r)\Big)\phi(r) &= 0,\\
-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}f_{NA}(r) + g^2\left(\frac{1}{2}\phi_1(r)^2 - \frac{1}{2}\phi(r)^2\right) &= 0.\\
-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}f(r) + 3e^2\left(\phi_1(r)^2 + 2\phi(r)^2 - 3\xi\right) &= 0.
\end{aligned}
$$

An individual vortex respects SU(2)xU(1), yielding
a four-parameter family of vortex solutions of equal tension

$$
q^{kA} = U \begin{pmatrix} e^{i\varphi}\phi_1(r) & 0 & 0 \\ 0 & \phi_2(r) & 0 \\ 0 & 0 & \phi_2(r) \end{pmatrix} U^\dagger,
$$

$$
A_i = U A_i^{(1,0,0)} U^\dagger,
$$

living on $$\frac{SU(3)}{SU(2)\times U(1)} \sim \mathrm{CP}^2.$$

Orientational zero modes

Generalization to U(N) theory straightforward:

A priori need 2N profile functions $\phi_1, \ldots, \phi_N, \quad f_3, \ldots, f_{N^2-1}, \quad f,$

$$q^{kA} = \begin{pmatrix} e^{in_1\alpha}\phi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{in_N\alpha}\phi_N \end{pmatrix},$$

$$A_i^3(x) = -\epsilon_{ij}\,\frac{x_j}{r^2}\Big((n_1 - n_2) - f_3\Big),$$

$$\vdots$$

$$A_i^{N^2-1}(x) = -\sqrt{\frac{2}{N(N-1)}}\epsilon_{ij}\,\frac{x_j}{r^2}\Big((n_1 + \ldots + n_{N-1} - (N-1)n_N) - f_{N^2-1}\Big),$$

$$A_i(x) = -\frac{1}{N}\epsilon_{ij}\,\frac{x_j}{r^2}\Big((n_1 + \ldots + n_N) - f\Big).$$

But for k=1 e.g. (0,0,..,1), vortex, can reduce to four profile functions by

$$\phi_1 = \ldots = \phi_{N-1} = \phi,$$

$$f_3 = \ldots = f_{(N-1)^2-1} = 0, \;\; f_{N^2-1} = -(N-1)f_{NA}$$

Each vortex solution respects U(N-1)

⇨ 2N-parameter family of degenerate vortices on $\dfrac{SU(N)}{SU(N-1)\times U(1)} \sim CP^{N-1}$

In terms of the moduli-matrix:

$$H = S^{-1}\, H_0(z), \qquad A_1 + iA_2 = -2\, i\, S^{-1}(z,\bar z)\, \bar\partial_z\, S(z,\bar z)$$

$$H_0 = N \times N \ \text{matrix, holomorphic in z}$$

k=1 vortex in the (1,0,..,0) patch

$$H_0^{(1,0,\dots,0)} = \begin{pmatrix} z - z_0 & 0 & 0 & \dots & 0 \\ b_1 & 1 & 0 & \dots & 0 \\ b_2 & 0 & \ddots & & 0 \\ \vdots & 0 & \dots & & 0 \\ b_{N-1} & 0 & 0 & \dots & 1 \end{pmatrix}$$

(b_1 , b_2 , ... b_{N-1}) ~ local coordinates of CP^{N-1}

Non-Abelian orientational modes of U(N) vortices

$$\langle q \rangle = \langle \tilde{q}^\dagger \rangle = v_2 \, \mathbb{1} = v_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \text{vortices}$$

Color-flavor locked vacuum

U ⊂ SU(N)

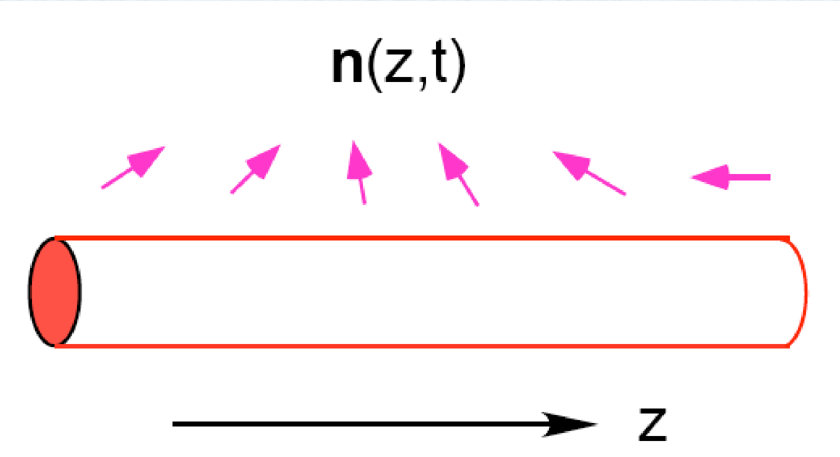
$$\underset{\text{color}}{q = U} \begin{pmatrix} e^{i\phi}\,\phi(r) & 0 & \dots & 0 \\ 0 & \chi(r) & 0 & \vdots \\ \vdots & 0 & \chi(r) & 0 \\ 0 & \dots & 0 & \ddots \end{pmatrix} \overset{\text{flavor}}{U^\dagger} \xrightarrow[r=0]{} U \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & w & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & w \end{pmatrix} U^\dagger$$

Exact $SU(N)_{C+F}$ group

Broken to U(N-1) by the soliton vortex (“Nambu-Goldstone modes”)

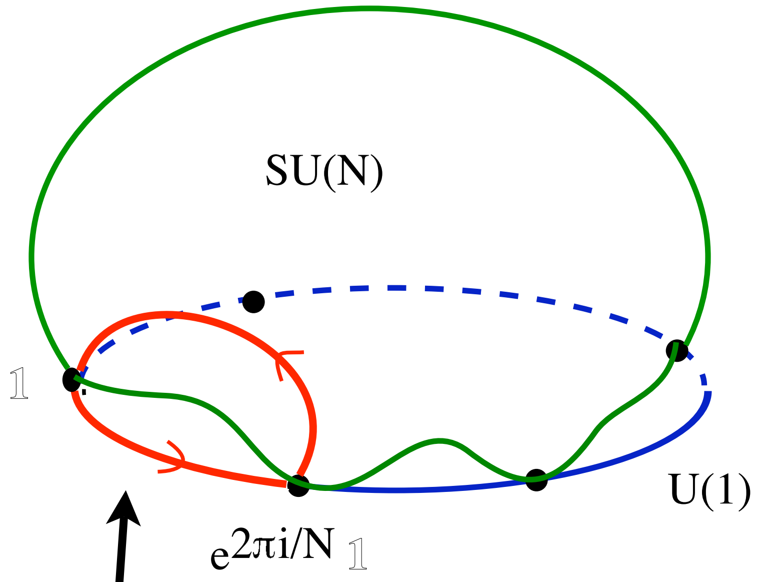
Vortex moduli = SU(N)/ U(N-1) = CP^{N-1} (= CP^1 ~ S^2 for U(2))

living only inside the vortex
(orientational zero modes)



Auzzi-Bolognesi-
Evslin-Konishi-Yung,
Hanany-Tong,
Shifman-Yung

Topological stability of non-Abelian vortices



Minimum noncontractible loop in the group (color) space SU(N)xU(1)/Z_N

Π_1 (SU(N)xU(1)/Z_N)=Z but the U(1) “charge” is 1/N

Intermezzo: supersymmetry

- Models with color-flavor locked vacuum natural (N=2 susy QCD)
- Supersymmetry ➪ self-dual vortices ($\lambda=g^2/2$)
- Non-renormalization theorem: the form of the potential protected from renormalization
- Dynamics under better control
- Physics depends on the parameter (e.g. masses) analytically (vortex vs monopoles)

$$\delta\psi_\alpha = i\sqrt{2}(\sigma_\mu)_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha} D^\mu q + \sqrt{2}(\sigma_\mu)_{\alpha\dot\alpha}\bar\xi^{\dot\alpha} D^\mu \bar{\tilde q}$$

$$\delta\lambda^\alpha = i(\sigma^{\mu\nu})^\alpha_\beta \epsilon^\beta F_{\mu\nu} + iD\,\epsilon^\alpha + \dots \qquad \Big(D = \sqrt{2}\,(\bar q q - c) \Big)$$

ε^α , ξ^α , $\alpha=1,2$, and c.c are the parameters of $\mathcal{N}=2$ susy

By setting $i\,\tau_3\,\xi = \varepsilon$, the above becomes

$$F_{12} = \sqrt{2}(\bar q q - c)$$

$$(D_1 + iD_2)\,q = 0$$

self-dual vortex equations ($\mathcal{N}$ = (2,2) supersymmetric)

- half of supersymmetry broken by vortex ➪ fermion zeromodes

Effective 2D sigma model

dt dz

- Allow for the dependence U= U(z,t) : the orientation fluctuates
- Vortex 2D dynamics in Higgs phase (U(2))

$$S_{\sigma}^{(1+1)} = \beta \int d^2x \, \frac{1}{2} \left(\partial \, n^a\right)^2$$ + fermionic terms

N=(2,2) supersymmetric CP^1 sigma model :
strongly coupled at low-energies
2 vacua ➞ kinks = (Abelian) monopoles !

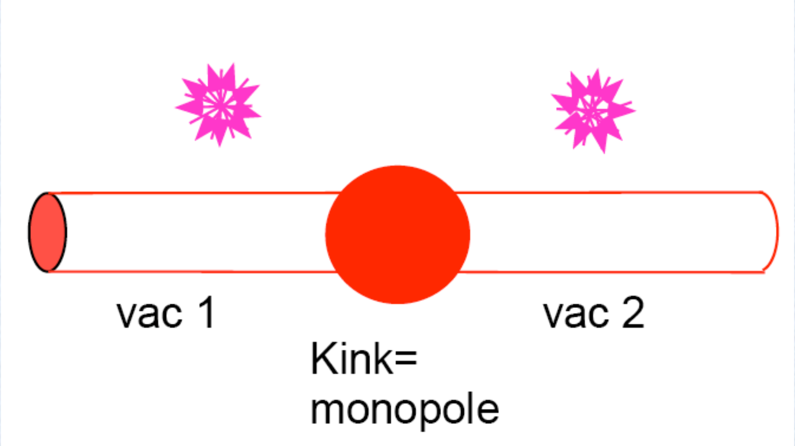
Vafa, Hori
ABEKY, Shifman-Yung

- ≡ Gauge dynamics in 4D in Coulomb phase

Tong, Shifman-Yung

(Seiberg-Witten)

- 2D - 4D duality Dorey
- Global SU(2) unbroken (Coleman)
- Vortex dynamically Abelianizes



Summary (up to this point):

U(N) gauge theory with N_f =N squarks fields:

Higgs mechanism

- SU(N)xU(1) gauge group broken completely

topological stability

- Π_1 (SU(N)xU(1)/Z_N)=Z ⇨ vortex

exact symmetry

- $SU(N)_{C+F}$ flavor symmetry unbroken $U \langle H \rangle U^{-1} = \langle H \rangle$ *
- Each vortex breaks the $SU(N)_{C+F}$ - flavor symmetry to U(N-1)

symmetry broken by soliton and vortex moduli

⇨ The orientational zeromodes on CP^{N-1} = SU(N)/U(N-1)

- The moduli-matrix formalism: tool for studying the vortex Moduli Space

techniques

- The orientational zeromodes may fluctuate along (z,t) : effective 2D supersymmetric CP^{N-1} sigma model. Dynamically Abelianize to $U(1)^N$

vortex quantum dynamics

⇨ • Search for systems with non-Abelian vortices of a more general types

supersymmetry

*) System with "color-flavor locked phase" appears naturally and automatically in $\mathcal{N}=2$ supersymmetric Quantum Chromodynamics (SQCD), in the equal mass case

Generalizations ('04-'09)

- Higher-winding vortices ($k>1$), the vortex moduli space } below
- $N_f > N$ systems: semi-local vortices; the vortex moduli space
- Systems with a non BPS (non self-dual) term : vortex interactions
- Stability of non BPS (non self-dual) semi-local vortices
- Non-Abelian vortices in Chern-Simons and Chern-Simons-YM systems in 2+1 D (S.B.Gudnason, 2009, 2010)

} no time

- Non-Abelian vortices in U(N) gauge theory, with product moduli space e.g., $CP^n \times CP^r$, $N= n+r$
- Non-Abelian vortices in more general class of gauge theory, $G= G' \times U(1)$, $G' = SO(N), USp(2N), \dots$

} next lecture

- Fractional vortices

Relevance to the non-Abelian monopoles } below

Generalizations '06-'07

EAH model
Vachaspati-Achucarro,
Hindmarsh

- Semilocal vortices (U(N) with N < N_f)

Tong, Shifman-Yung,
Eto-Evslin-KK-Marmorini-Nitta-Ohashi-Vinci-Yokoi

New (Seiberg-like) duality

N_f=3,N_c=2

N_f=3,N_c=1

$$\begin{tikzcd}
& WCP^2[1,1,-1] \arrow[dl] \arrow[dr] & \\
\tilde{\mathbf{C}}^2 \arrow[rr, dotted, leftrightarrow, "\text{Seiberg duality}"] \arrow[dr, "g^2\to\infty"'] & & \mathbf{C}^2 \arrow[dl, "\tilde{g}^2\to\infty"] \\
& (\mathbf{C}^2)^* &
\end{tikzcd}$$

- Non-BPS vortices

Nontrivial interactions among the vortices

Type III vortices

Shifman-Yung;
Auzzi-Eto-Vinci;
Gudnason-Bolognesi

- Vortices in SO(N)xU(1) theories

GNO duality :

SO(2N+1) ⇔ USp(2N);

SO(2N) ⇔ $\widetilde{\text{SO(2N)}}$, etc

Ferretti-Gudnason-KK
'07

Eto-Fujimori-Gudnason-KK-
Nagashima-Nitta-Ohashi
'08

Further generalizations: ’08-’09

- Non-BPS Non-Abelian vortices: stability

Auzzi-Eto-Gudnason-KK-Vinci
‘08

- Non-Abelian vortices with product moduli (no dynamical Abelianization)

Dorigoni-KK-Ohashi
‘08

- General gauge groups

Vortex for G= G’ x U(1): arbitrary G’

Eto-Fujimori-Gudnason-KK-Nagashima-Nitta-Ohashi
‘08

- Fractional vortices

Vortices with higher winding numbers '06 -

Detailed study of k=2 (axially symmetric) vortices of U(N) theory

Hashimoto-Tong;
Auzzi-Shifman-Yung;
Pisa-TiTech '06-07

$\square \otimes \square = \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$ under $SU(N)_{C+F}$

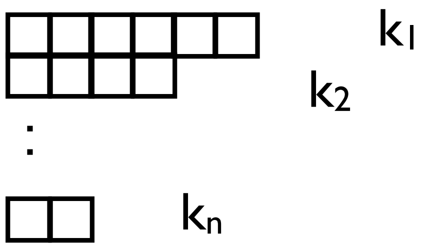
For U(2), k=2 case: the vortex moduli space = $WCP_{(2,1,1)}$ (➔ next pages)

Vortices for generic k in U(N) theory

Pisa-Tokyo-Kyoto '09
(preliminary)

transform as

$\underbrace{\square \otimes \square \otimes \square \otimes \ldots}_{k} = \sum \oplus$



Sum of the Young tableaux ~ Irreps of SU(N)

Points in classical vortex moduli space transform as quantum mechanical states in various (in general reducible) representations of SU(N)

Higher-winding vortices

k=2 vortex moduli SU(2)

(z = x + i y)

$$H_0^{(1,0)}(z) = \begin{pmatrix} z - z_0 & 0 \\ -b' & 1 \end{pmatrix}, \qquad H_0^{(0,1)}(z) = \begin{pmatrix} 1 & -b \\ 0 & z - z_0 \end{pmatrix}$$

$$b = \frac{1}{b'}$$ CP^1

k=1 vortex

Moduli matrix in three local patches (related by V transf.)

$$H_0^{(2,0)} = \begin{pmatrix} z^2 & 0 \\ -a'\,z - b' & 1 \end{pmatrix}, \; H_0^{(1,1)} = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z + \phi \end{pmatrix}, \; H_0^{(0,2)} = \begin{pmatrix} 1 & -a\,z - b \\ 0 & z^2 \end{pmatrix}$$

$$X\,Y \equiv -\phi, \quad X^2 \equiv \eta, \quad Y^2 \equiv -\tilde{\eta}. \qquad \phi^2 + \eta\,\tilde{\eta} = 0.$$

det H = z^2

co-axial

$$\begin{pmatrix} a' \\ 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} \sim \begin{pmatrix} -a \\ b \\ 1 \end{pmatrix}$$

$$\tilde{\mathcal{M}}_{N=2,k=2} \simeq WCP^2_{(2,1,1)} \simeq$$

$$\simeq CP^2/Z_2 \simeq CP^2$$

WCP$_{(2,1,1)}$: $(z_1, z_2, z_3) \sim (\lambda^2 z_1, \lambda z_2, \lambda z_3)$, $\lambda \in C^*$

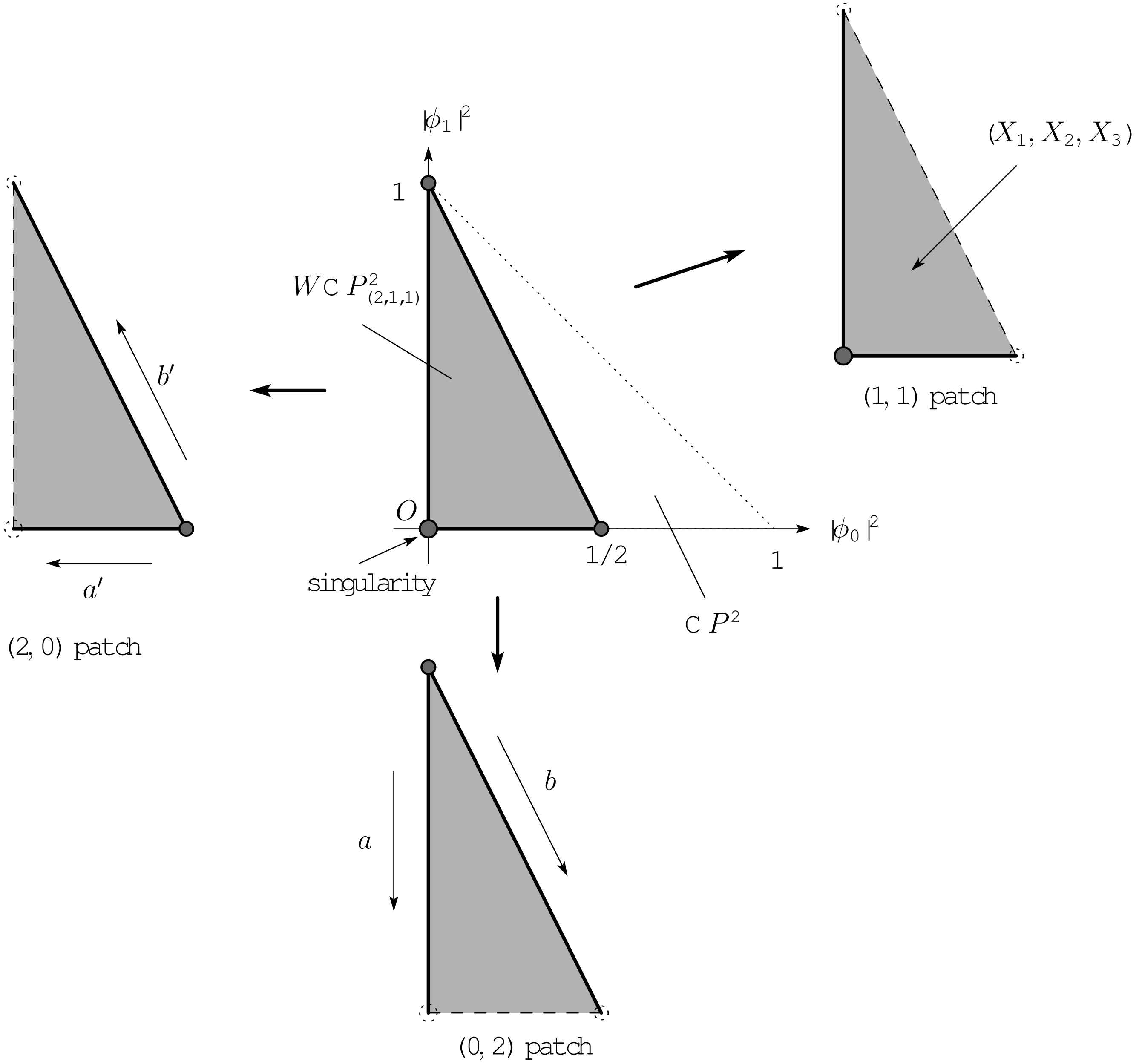
$$\mathcal{M}^{\text{separated}}_{k=2,N=2} \simeq \left(\mathbf{C} \times \mathbf{C}P^1\right)^2 / \mathfrak{S}_2,$$

Eto, Konishi,Marmorini,Nitta,Ohashi, Vinci,Yokoi '06

Hashimoto-Tong, Shifman-Yung '06

SO(5) ⇒ U(2) ⇒ 1 : k=2 vortices are in <u>3</u> + <u>1</u> of SU(2) ⇒

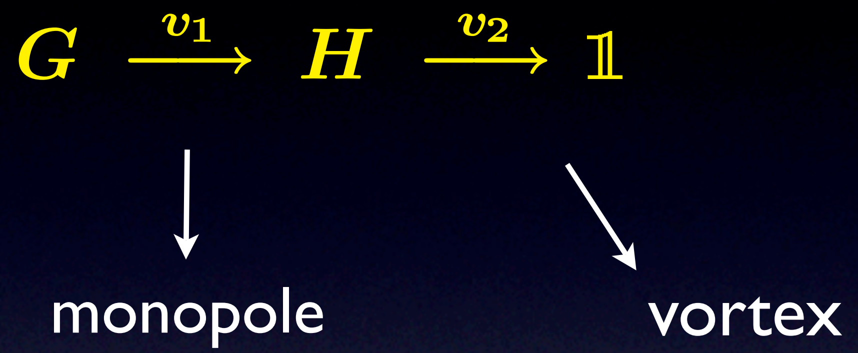
Monopoles (E.Weinberg) ~ <u>3</u> or <u>1</u> of SU(2)!



$|\phi_1|^2$
1
$WCP^2_{(2,1,1)}$
O
singularity
1/2
1
$|\phi_0|^2$
CP^2
(X_1, X_2, X_3)
(1, 1) patch
b'
a'
(2, 0) patch
b
a
(0, 2) patch

Monopole-vortex connection

Consider hierarchical symmetry breaking



$$G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1}$$

$$v_1 \gg v_2,$$

monopole

vortex

- Apparent paradox (no monopoles, no vortices) ⇨
- Topology and symmetry
  connect monopoles and vortices
- Non-Abelian vortices ⇨ non-Abelian monopoles

A 30-years old problem,
possibley relevant to
quark confinement

Homotopy-group map

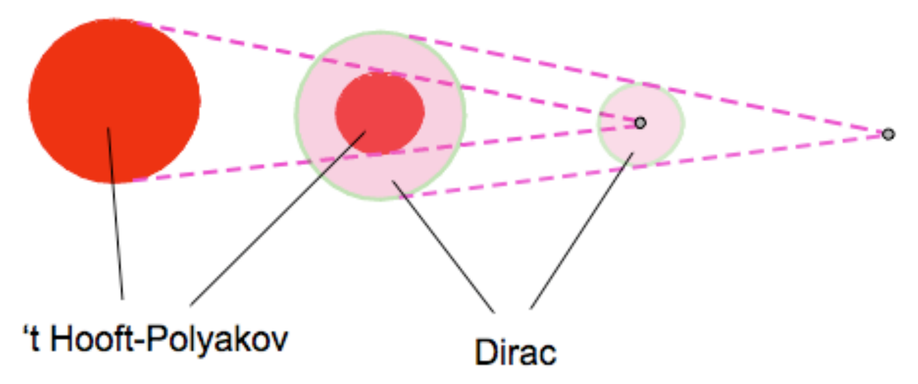
$$G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1}$$

$v_1 \gg v_2,$

Homotopy-group exact sequence:

Vortex ! (but also monopole)

$$\cdots \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \cdots$$



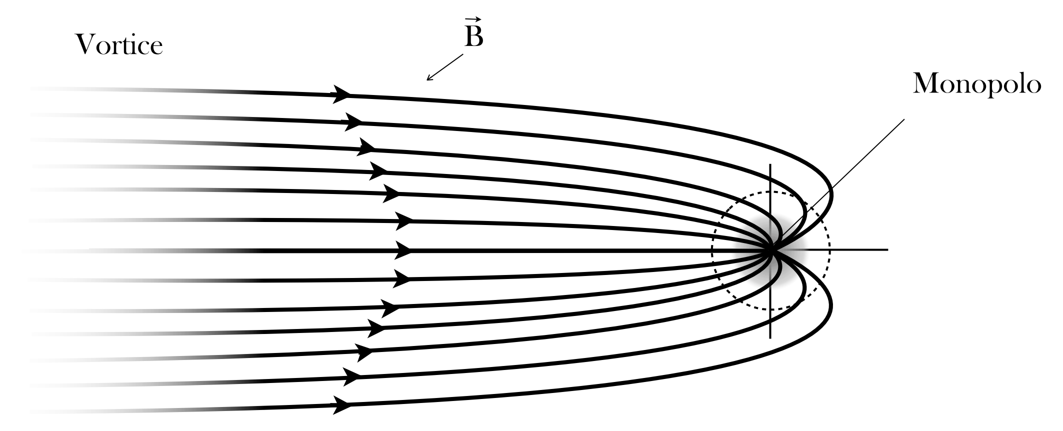
- $\pi_2(G) = 1$ ⇒ Regular monopoles confined by vortices

{
- $\pi_1(G) = 1$ ⇒ All vortices "end" at regular monopoles e.g. SU(N)
- $\pi_1(G) = Z_2$ ⇒ k=2 vortices "end" at regular monopoles!

't Hooft SO(3)/U(1)

cfr., SO(N) k=1 vortices are there: confine Dirac monopoles

Non-Abelian monopole moduli from vortex moduli in the system $G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1}$



Flux matching

(Auzzi-Bolognesi-Evslin-KK; Kneipp)

SU(N+1) ⇒ SU(N) x U(1)

⇒ 1

Exact H_{C+F} induces continuous transformation of vortex -- **and monopole**

Study in more detail this!

A tricky point

$$\begin{tikzcd} G \arrow[rr, "v_1"] & {} \arrow[d] & H \arrow[rr, "v_2"] & {} \arrow[dr] & \mathbb{1} \\ & \text{monopole} & & & \text{vortex} \end{tikzcd}$$

$$v_1 \gg v_2,$$

- Vortices of the low-energy theory ($v_1 = \infty$) are BPS
- Monopoles of the high-energy theory ($v_2 = 0$) are stable by $\Pi_2(G/H)$
- Together, they are not BPS, only approximately BPS
    - bad: the monopole-vortex complex is not a solution (not stable)
    - good: Non-Abelian vortices ⇨ non-Abelian monopoles; good: real mesons

$$E = \int d^3x \left[\frac{1}{4} F_{ij}^{a\,2} + \frac{1}{2} (D_i \phi^a)^2 + \frac{\lambda}{8} (\phi^{a\,2} - F^2)^2 \right]$$

$$= \int d^3x \left[\frac{1}{4} (F_{ij}^a - \epsilon_{ijk} D_k \phi^a)^2 + \frac{1}{2} \epsilon_{ijk} F_{ij}^a D_k \phi^a + \frac{\lambda}{8} (\phi^{a\,2} - F^2)^2 \right.$$

$$F_{ij}^a = \epsilon_{ijk} D_k \phi^a, \quad \frac{1}{2} \epsilon_{ijk} F_{ij}^a D_k \phi^a = \partial_k B_k, \qquad B_k = \frac{1}{2} \epsilon_{ijk} F_{ij}^a \phi^a.$$

Non-Abelian monopoles

Goddard-Nuyts-Olive, E.Weinberg, Lee,Yi,
Bais, Schroer, ... '77-80

$$G \xrightarrow{\langle\phi\rangle\neq 0} H$$

H: non-Abelian

$$F_{ij} = \epsilon_{ijk}\frac{r_k}{r^3}(\beta\cdot \mathrm{T}), \qquad 2\,\beta\cdot\alpha \in \mathbf{Z}$$

cfr. (Dirac)

$2\ m \cdot e \in Z$

"Monopoles are multiplets of $\widetilde{\mathrm{H}}$ (GNOW)"

β= weight vector of the group $\widetilde{\mathrm{H}}$ generated by $\alpha^* \equiv \frac{\alpha}{\alpha\cdot\alpha}.$

$<\Phi> = v_I = h \cdot T$

H	$\widetilde{\mathrm{H}}$
U(N)	U(N)
SU(N)	SU(N)/Z_N
SO(2N)/Z_2	$\widetilde{SO(2N)}$
SO(2N+1)	USp(2N)

$$SU(3) \xrightarrow{\langle\phi\rangle} \frac{SU(2)\times U(1)}{\mathbb{Z}_2}, \qquad \langle\phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}$$

Difficulties

① Topological obstructions (Abouelsaad et.al. '83)

Φ = diag(v,v,-2v)

↵

e.g., SU(3) → SU(2)×U(1),

∄ monopoles ~ $(2, 1^{*}\ )$

"No colored dyons exist" (Coleman, et.al. '84)

② Non-normalizable gauge zero modes: (Dorey, et.al. '96)

Monopoles not multiplets of H

cfr.
Jackiw-Rebbi
Flavor Q.N. of monopoles
via
fermion zeromodes

The real issue:
how do they transform under $\tilde{H}$?

- H and $\tilde{H}$ relatively nonlocal
- $\tilde{H}$ theory in confinement phase ⇔ H theory in Higgs phase

Light non-Abelian monopoles ('94-'00)

Fully quantum-mechanical non-Abelian monopoles in N=2 supersymmetric theories (also N=1, N=4)

Seiberg-Witten '94
Argyres,Plesser,Seiberg,'96
Hanany-Oz, '96
Carlino-KK-Murayama '00

Table

- Colored dyon ~ $(2^*, 1^*)$ in SU(3) → SU(2)×U(1) do exist !

- Non-Abelian dual groups (monopoles) only in theories with flavors
  - Renomalization-Group effect: the dual SU(r) group only for $r < N_f /2$
  - Only Abelian monopoles in pure *N*=2 YM or with SU(2) group

Softly broken N=2 supersymmetric SU, SO, USp

$$G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1} \qquad v_1 \gg v_2,$$

mass
parameters

G=SU(N+1); H=U(N)

$$\mathcal{L} = \frac{1}{8\pi} \mathrm{Im}\, S_{cl} \left[\int d^4\theta\, \Phi^\dagger e^V \Phi + \int d^2\theta\, \frac{1}{2} W W \right] + \mathcal{L}^{(quarks)} + \int d^2\theta\, \mu\, \mathrm{Tr}\Phi^2;$$

$$\mathcal{L}^{(quarks)} = \sum_i [\int d^4\theta\, \{ Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger \} + \int d^2\theta\, \{ \sqrt{2} \tilde{Q}_i \Phi Q^i + m \tilde{Q}_i Q^i \}$$

m ≫ μ ≫ Λ :
semi-classical

Bosonic Lagrangean

$$\mathcal{L} = \frac{1}{4g^2} F_{\mu\nu}^2 + \frac{1}{g^2} |\mathcal{D}_\mu \Phi|^2 + |\mathcal{D}_\mu Q|^2 + \left| \mathcal{D}_\mu \bar{\tilde{Q}} \right|^2 - V_1 - V_2,$$

m ~ μ ~ Λ :
fully quant. mech.

$$\langle \Phi \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \dots & m & 0 \\ 0 & \dots & 0 & -N\, m \end{pmatrix};$$

v_1 = m
v_2 = √μm

SU(N+1) ⇒ U(N)

$$Q = \tilde{Q}^\dagger = \begin{pmatrix} d & 0 & 0 & 0 & 0 & \dots \\ 0 & \ddots & 0 & \vdots & \vdots & \dots \\ 0 & 0 & d & 0 & 0 & \dots \\ 0 & \dots & 0 & -N\, d & 0 & \dots \end{pmatrix}, \qquad d = \sqrt{(N+1)\, \mu\, m} \ll m.$$

Phases of Softly Broken $\mathcal{N} = 2$ Gauge Theories

label (r)	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
0	monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f)$
1	monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f-1) \times U(1)$
$\leq [\frac{n_f-1}{2}]$	NA monopoles	$SU(r) \times U(1)^{n_c-r}$	Confinement	$U(n_f-r) \times U(r)$
$n_f/2$	rel. nonloc.	-	Confinement	$U(n_f/2) \times U(n_f/2)$
BR	NA monopoles	$SU(\tilde{n}_c) \times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$U(n_f)$

Table 1: Phases of $SU(n_c)$ gauge theory with n_f flavors. $\tilde{n}_c \equiv n_f - n_c$.

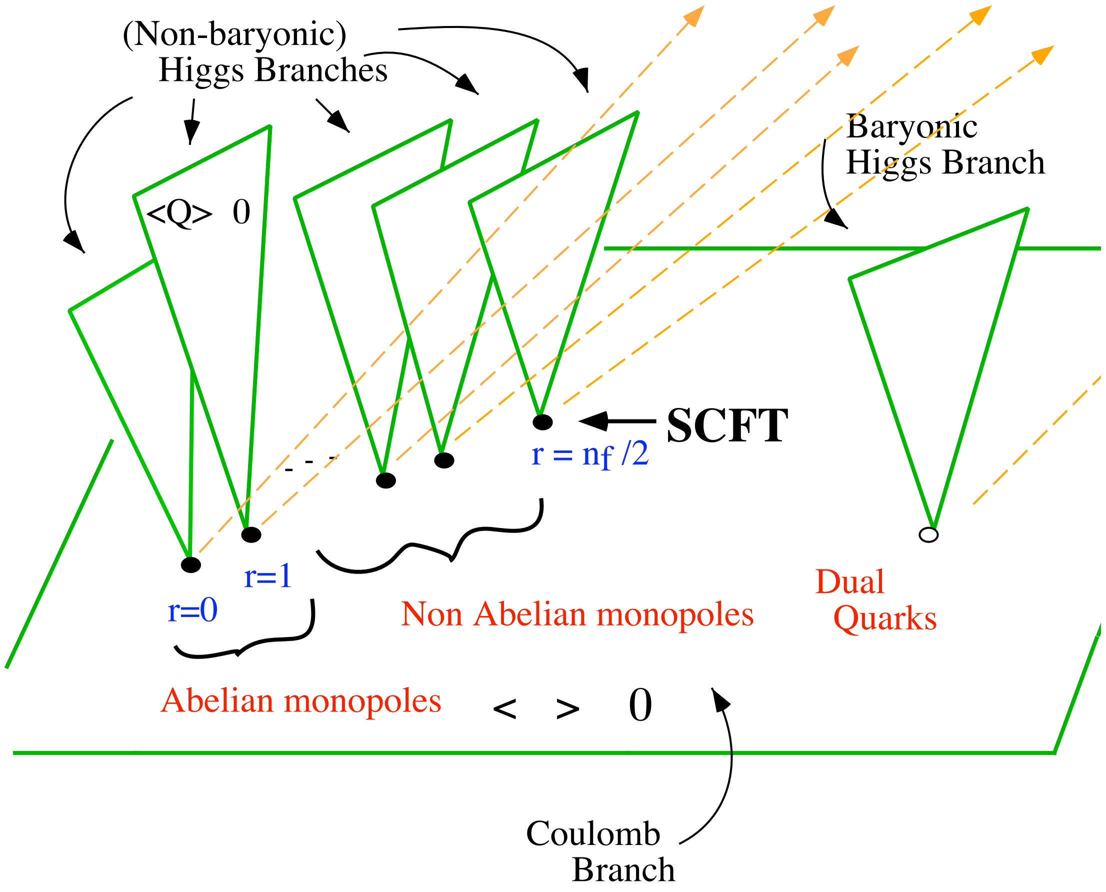
	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
1st Group	rel. nonloc.	-	Confinement	$U(n_f)$
2nd Group	dual quarks	$USp(2\tilde{n}_c) \times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$SO(2n_f)$

Table 2: Phases of $USp(2n_c)$ gauge theory with n_f flavors with $m_i \to 0$. $\tilde{n}_c \equiv n_f - n_c - 2$.

$$\mathcal{W}(\phi, Q, \tilde{Q}) = \mu \, \mathrm{Tr}\Phi^2 + m_i \tilde{Q}_i Q^i, \qquad m_i \to 0$$

Dual qualks of r vacua are GNO monopoles

QMS of N=2 SQCD (SU(n) with n_f quarks)



● N=1 Confining vacua (with Φ^2 perturbation)

○ N=1 vacua (with Φ^2 perturbation) in free magnetic pha

Summary: Lecture I

- Non-Abelian vortices in U(N) gauge theory with $N_f = N$ matter fields
- Color-flavor locked vacuum
- Vortex moduli in CP^{N-1}
- Supersymmetry: marginal role classically, but more important in the dynamics. Self-dual equations

Generalization:

- Vortices in general gauge systems
- Vortices with product moduli space
- Fractional vortices
- Monopole-vortex complex

⇨ Lecture II

Lecture II

- Vortices in general gauge systems
- Vortices with product moduli space
- Fractional vortices
- Monopole-vortex complex

§ 1. Vortex in general gauge theories

('08,'09) M. Eto, M. Nitta, S.B. Gudnason, W. Vinci,
K.K. T. Fujimori, T. Nagashima, K.Ohashi
(Pisa, Tokyo, Cambridge)

- G= U(1) x G'
- G'= SU(N), SO(N), USp(N), ...
- FI term for the U(1) factor (vortex)
- Color-flavor locked phase with exact, unbroken G' $_{C+F}$ symmetry

$$\mathcal{L} = \mathrm{Tr}_c\left[-\frac{1}{2e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \mathcal{D}_\mu H\left(\mathcal{D}^\mu H\right)^\dagger - \frac{e^2}{4}\left|X^0t^0 - 2\xi t^0\right|^2 - \frac{g^2}{4}\left|X^at^a\right|^2\right]$$

$$X = HH^\dagger = X^0t^0 + X^at^a + X^\alpha t^\alpha \ ,$$

$$\langle H\rangle = \frac{v}{\sqrt{N}}1_N \ , \qquad \xi = \frac{v^2}{\sqrt{2N}}$$

$$H = S^{-1}(z,\bar{z})H_0(z) \ , \qquad \bar{A} = -iS^{-1}(z,\bar{z})\bar{\partial}S(z,\bar{z})$$

$$\bar{\mathcal{D}}H = \bar{\partial}H + i\bar{A}H = 0 \ ,$$
$$F_{12}^0 = e^2\left[\,\mathrm{Tr}_c\left(HH^\dagger t^0\right) - \xi\,\right] \ ,$$
$$F_{12}^a = g^2\,\mathrm{Tr}_c\left(HH^\dagger t^a\right) \ .$$

$H_0(z)$ moduli matrix

General Procedure (SO(2M), USp(2M), SO(2M+1))

Self-dual equations

$$\mathcal{D}_{\bar{z}} H = 0, \quad (*)$$

$$F^0_{12} - \frac{e^2}{\sqrt{2N}} \left(\mathrm{tr}\,(HH^\dagger) - v^2\right) = 0,$$

$$F^a_{12} t^a - \frac{g^2}{4} \left(HH^\dagger - J^\dagger (HH^\dagger)^T J\right) = 0, \quad (**)$$

Matter equation (*)
solved by the Ansatz

$$W_1 + iW_2 = -2iS^{-1}(z,\bar{z})\bar{\partial} S(z,\bar{z})$$

$$H = S^{-1} H_0(z) = S_e^{-1} S'^{-1} H_0(z)$$

$$S(z,\bar{z}) = S_e(z,\bar{z}) S'(z,\bar{z}) \qquad S_e \in U(1)^{\mathbb{C}} \simeq \mathbb{C}^*$$

H_0 , S defined up to

$$(H_0, S) \sim V_e V'(z)(H_0, S), \quad V'(z)^T J V'(z) = J.$$

Define now

$$\Omega_e \equiv S_e S_e^\dagger \equiv e^{\psi 1_{2N}} \in U(1)^{\mathbb{C}}, \quad \Omega' \equiv S' S'^\dagger \in G'^{\mathbb{C}},$$

$$\Omega_0 \equiv H_0 H_0^\dagger$$

Gauge-field equations
become (master eq.)
given H_0 (z),

$$\bar{\partial}\partial\psi = -\frac{e^2}{4N}\left(\mathrm{tr}\,(\Omega_0 \Omega'^{-1}) e^{-\psi} - v^2\right),$$

$$\bar{\partial}(\Omega' \partial \Omega'^{-1}) = \frac{g^2}{8}\left(\Omega_0 \Omega'^{-1} - J^\dagger (\Omega_0 \Omega'^{-1})^T J\right) e^{-\psi},$$

(**) $$J = \begin{pmatrix} 0_M & 1_M \\ \epsilon 1_M & 0_M \end{pmatrix} \qquad \epsilon = \pm 1, \quad SO(2M), USp(2M)$$

Holomorphic Invariants

$$I^i_{G'}(H) = I^i_{G'}\left(s^{-1}S'^{-1}H_0\right) = s^{-n_i} I^i_{G'}(H_0(z))$$

G' - invariants made of H

n_i U(1) charge

I(H)= det H
for U(N)

Also I(H)=$H^{\rm T}JH$

for SO(2M), USp(2M)

$$J = \begin{pmatrix} 0_M & 1_M \\ \epsilon 1_M & 0_M \end{pmatrix}$$

$$I^i_{G'}(H)\Big|_{|z|\to\infty} = I^i_{\rm vev}\, e^{i\nu n_i\theta}$$

$$I^i_{G'}(H_0) = s^{n_i} I^i_{G'}(H) \xrightarrow{|z|\to\infty} I^i_{\rm vev}\, z^{\nu n_i} .$$

$$\nu\, n_i \in \mathbb{Z}_+ \qquad \to \qquad \nu = \frac{k}{n_0}\,, \qquad k \in \mathbb{Z}_+$$

➔ H_0 (z)

$$n_0 \equiv \gcd\left\{n_i \,|\, I^i_{\rm vev} \neq 0\right\}$$

$$G = \left[U(1) \times G'\right]/\mathbb{Z}_{n_0}$$

U(1) winding #

= 2 (SO(2N), USp(2N))
= 1 SO(2N+1) ; = N for SU(N)

GNOW (Goddard-Nuyts-Olive-E. Weinberg) quantization

Representative (vortex) solutions

$$H_0(z) = z^{\nu 1_N + \nu_a \mathcal{H}_a} \in U(1)^{\mathbb{C}} \times G'^{\mathbb{C}} ,$$

$$(\nu 1_N + \nu_a \mathcal{H}_a)_{ll} \in \mathbb{Z}_{\geq 0} \quad \forall\, l$$

$\mathcal{H}_a$ = Cartan subalgebra of G'

$\longrightarrow$ $\nu + \nu_a \mu_a^{(i)} \in \mathbb{Z}_{\geq 0} \quad \forall\, i$

$\vec{\mu}^{(i)} = \mu_a^{(i)}$ weight vectors of G'

$\longrightarrow$ $\vec{\nu} \cdot \vec{\alpha} \in \mathbb{Z}$ (❣)

$\vec{\alpha}$ root vectors of G'

$\longrightarrow$ Solution: $\tilde{\vec{\mu}} \equiv \vec{\nu}/2$ is a weight vector of $\tilde{G}'$

Remarks :

- (❣) formally identical to the GNOW "quantization" for the monopoles (Goddard-Nuyts-Olive, E. Weinberg)
- (❣) formally identical to that found for "non-Abelian vortices" for YM (Spanu-Konishi)
- The latter are actually Z_N vortices
- The former has the well-known difficulties
- Our vortices have continuous (orientational) moduli
- Their transformation ~ various irred. representations of the dual G' group, $\tilde{G}'$
- Explicitly checked with G'= SU(N), SO(2N); Other groups under study

Vortex in SO(2N)xU(1)/Z_2 models

Gudnason-Ferretti-KK

$$q(r,\vartheta)=\begin{pmatrix} \mathbf{M}_1(r,\vartheta) & 0 & 0 & \cdots \\ 0 & \mathbf{M}_2(r,\vartheta) & 0 & \cdots \\ 0 & 0 & \mathbf{M}_3(r,\vartheta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

M_i ~ 2x2 matrices

$$H^{(a)}=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2a+1,2a+2}$$

Squark fields at large r = SO(2N)xU(1)/Z_2
closed (non-contractible) gauge orbits

$$q(\varphi)\sim e^{i[\frac{1}{2}T_0+\sum_i(\pm\frac{1}{2})T_i]\varphi}$$

Minimum vortices classified by the
$U_0(1)$ and Cartan U(1) charges

$$\mathcal{V}\ \sim\ \mathrm{SO(2N)/U(N)}$$

~ 2^{N-1} dim spinor representations
of an SO(2N)

Each of them leaves an U(N) ⊂
$\mathrm{SO(2N)}_{C+F}$ unbroken

Vortex moduli space ~ quantum states of a particle in
2^{N-1} dim spinor repr.

Examples: k=1 vortices for G' = SO(2N) and USp(2N)
Moduli matrices

skew-diagonal basis
Q^T J Q = inv,

J= $\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \pm\mathbf{1} & \mathbf{0} \end{pmatrix}$

$$H_0(z) = \begin{pmatrix} z\,\mathbf{1}_{N\times N} & \mathbf{0} \\ \mathbf{B} & \mathbf{1}_{N\times N} \end{pmatrix}$$

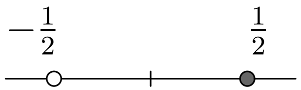
Complex matrix B, are symmetric or antisymetric
B^T = B, - B for USp(2N), SO(2N), respectively

Complex matrix B contain

$\frac{N(N+1)}{2}$ free (complex) parameters labeling the coset USp(2N)/U(N)

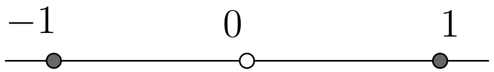
$\frac{N(N-1)}{2}$ free (complex) parameters labeling the coset SO(2N)/U(N)

Elements of B are the local coordinates

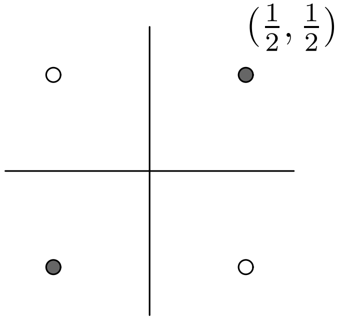


$SO(2)$

$USp(2)$

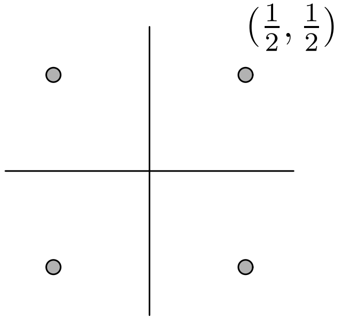


$SO(3)$



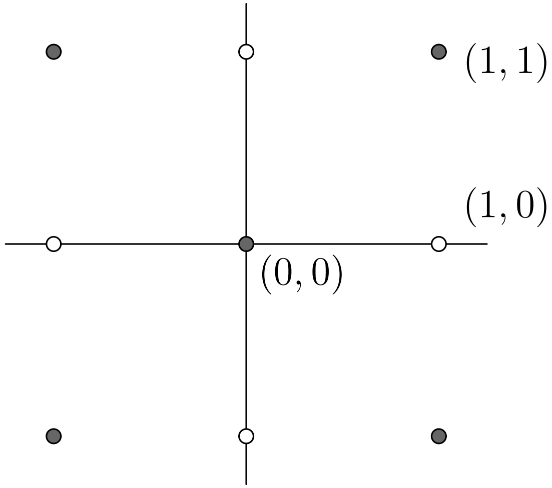
$SO(4)$

$(\leftrightarrow \widetilde{SO}(4) \sim SU(2) \times SU(2))$

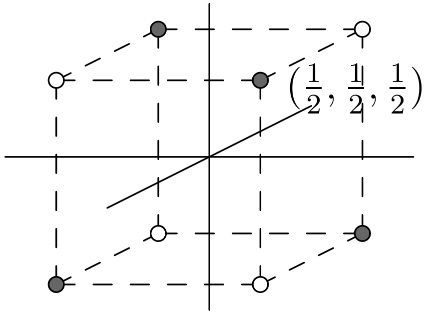


$USp(4)$

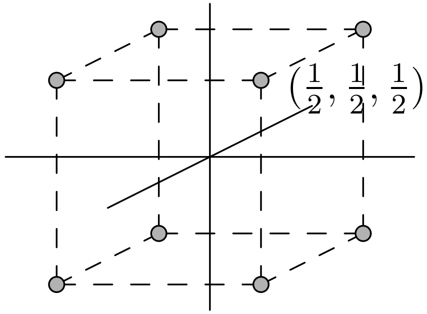
$(\leftrightarrow \widetilde{SO}(5))$



$SO(5) \quad (\leftrightarrow USp(4))$



$SO(6) \ (\leftrightarrow \widetilde{SO}(6))$



$USp(6) \ (\leftrightarrow \widetilde{SO}(7))$

k=1

vortices

§ 2. Vortex with product moduli

Dorigoni-Ohashi-Konishi '09

- The non-Abelian vortex in U(N) theory with $N_f = N$ (*) dynamically Abelianizes
- Correspondence classical-quantum r vacua in fact suggests that the orignal "non-Abelian vortex" (*) is related to the quantum r=0 vacuum (with Abelian monopoles)
- In 4D $\mathcal{N}=2$ Supersymmetric QCD, there are vacua with light non-Abelian monopoles
- There must be, in semi-classical region, corresponding vortices which do not completely Abelianize

4D: • U(N) low-energy model from SU(N+1) ⇒ SU(N) × U(1)/Z

- r= N_f vacuum (classical)

$$\langle\Phi\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \dots & m & 0 \\ 0 & \dots & 0 & -N\,m \end{pmatrix};$$

- quantum mechanically only r < N_f / 2
- classical (r) ⇔ quantum (N_f - r) vacua

m ≫ μ ≫ Λ : ⇔ m ~ μ ~ Λ :

(Vacuum counting; symmetry)

- U(N) model : quantum r =0 vacua ! (Abelian monopoles only)

N.B.

r	Deg. Freed.	Eff. Gauge Group	Phase	Global Symmetry
0	monopoles	$U(1)^{N-1}$	Confinement	$U(n_f)$
1	monopoles	$U(1)^{N-1}$	Confinement	$U(N_f-1)\times U(1)$
$2, .., [\frac{N_f-1}{2}]$	NA monopoles	$SU(r)\times U(1)^{N-r}$	Confinement	$U(N_f-r)\times U(r)$
$N_f/2$	rel. nonloc.	-	Almost SCFT	$U(N_f/2)\times U(N_f/2)$

Q: Non-Abelian vortices which do not dynamically Abelianize ?

(&) U(N) model (with N_f =N “flavors” of complex scalar fields -- squarks)

$$\mathcal{L} \;=\; \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \frac{2}{g^2}\mathcal{D}_\mu\,\phi^\dagger\,\mathcal{D}^\mu\phi - \mathcal{D}_\mu\,H\,\mathcal{D}^\mu H^\dagger - \lambda\left(c\,1_N - H\,H^\dagger\right)^2\right]$$
$$+\;\; \mathrm{Tr}\;[\,(H^\dagger\phi - M\,H^\dagger)(\phi\,H - H\,M)\,]$$

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\nu + i\,[W_\mu, W_\nu] \text{ and } \mathcal{D}_\mu H = (\partial_\mu + i\,W_\mu)\;H,$$

$(H)^i_\alpha \equiv q^i_\alpha$: N complex scalar fields in the fundamental representation of SU(N), written in color-flavor mixed matrix form

ϕ A complex scalar field in the adjoint representation of SU(N)

$M = diag\,(m_1, m_2, \ldots, m_N)$ is the mass matrix for the squarks q

- For a critical coupling constant $\lambda = \dfrac{g^2}{4}$ *) BPS (self-dual) (automatic in Susy)

the model can be regarded as a truncation of the bosonic sector of a N=2 supersymmetric model, with $(H)^i_\alpha \equiv q^i_\alpha, \quad \tilde{q}^\alpha_i \equiv 0$

- In this case c comes from the Fayet-Iliopoulos term L = c V|D

- For unequal masses

$$\langle\phi\rangle = M = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & m_N \end{pmatrix}$$

breaks U(N) ➔ U(1)^N

U(1), s broken by the squark vac. exp. value ➔ ANO vortex nothing really new

The Model: the same SU(N), N =N_f, softly broken N=2 SQCD (&) but with appropriately tuned masses

N=1 SQCD

QMS ⇔ m_i ,

$m_i \Rightarrow 0$

$$M = \begin{pmatrix} m^{(1)}\,\mathbb{1}_{n\times n} & 0 \\ 0 & m^{(2)}\,\mathbb{1}_{r\times r} \end{pmatrix}$$

$$N = n + r \; ;$$

i.e., $$\boxed{n\, m^{(1)} + r\, m^{(2)} = 0}$$

or $$m^{(1)} = \frac{r\, m_0}{\sqrt{r^2+n^2}}, \quad m^{(2)} = -\frac{n\, m_0}{\sqrt{r^2+n^2}},$$

$$|m_0| \gg |\mu| \gg \Lambda \, .$$

Adjoint scalar VEV $$\langle \Phi \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m^{(1)}\,\mathbb{1}_{n\times n} & 0 \\ 0 & m^{(2)}\,\mathbb{1}_{r\times r} \end{pmatrix}$$

SU(N)$|_{C+F}$ ⇒ $$G = \frac{SU(n)\times SU(r)\times U(1)}{\mathbb{Z}_K}, \quad K = \mathrm{LCM}\,\{n, r\}$$

$$Q(x) = \begin{pmatrix} q^{(1)}(x)_{n\times n} & 0 \\ 0 & q^{(2)}(x)_{r\times r} \end{pmatrix},$$

$$\tilde{Q}(x) = \begin{pmatrix} \tilde{q}^{(1)}(x)_{n\times n} & 0 \\ 0 & \tilde{q}^{(2)}(x)_{r\times r} \end{pmatrix}$$

fields	$U(1)$	$SU(n)$	$SU(r)$
$q^{(1)}$	λ_1	$\underline{n}$	$\underline{1}$
$\tilde{q}^{(1)}$	$-\lambda_1$	$\underline{n}^*$	$\underline{1}$
$q^{(2)}$	$-\lambda_2$	$\underline{1}$	$\underline{r}$
$\tilde{q}^{(2)}$	λ_2	$\underline{1}$	$\underline{r}^*$

$$\lambda_1 \equiv \frac{r}{\sqrt{2\,n\,r\,(r+n)}}; \qquad \lambda_2 \equiv \frac{n}{\sqrt{2\,n\,r\,(r+n)}}$$

vortex Ansatz:
V_D=0

$$\longleftarrow \left\{ \begin{array}{l} \tilde{q}^{(1)} = (q^{(1)})^\dagger, \qquad q^{(2)} = -(\tilde{q}^{(2)})^\dagger\,; \\ q^{(1)} \to \frac{1}{\sqrt{2}}\,q^{(1)}, \quad \tilde{q}^{(2)} \to \frac{1}{\sqrt{2}}\,\tilde{q}^{(2)} \end{array} \right.$$

$$\langle Q\rangle = \begin{pmatrix} v^{(1)}\,\mathbb{1}_{n\times n} & 0 \\ 0 & -v^{(2)\,*}\,\mathbb{1}_{r\times r} \end{pmatrix}, \quad \langle\tilde{Q}\rangle = \begin{pmatrix} v^{(1)\,*}\,\mathbb{1}_{n\times n} & 0 \\ 0 & v^{(2)}\,\mathbb{1}_{r\times r} \end{pmatrix},$$

breaks G completely

$$|v^{(1)}|^2 + |v^{(2)}|^2 = \sqrt{\frac{n+r}{n\,r}}\,\mu\,m_0$$

$$\mathcal{L} = -\frac{1}{4g_0^2}F^{0\,2}_{\mu\nu} - \frac{1}{4g_n^2}F^{n\,2}_{\mu\nu} - \frac{1}{4g_r^2}F^{r\,2}_{\mu\nu} + \frac{1}{g_0^2}|\mathcal{D}_\mu\Phi^{(0)}|^2 + \frac{1}{g_n^2}|\mathcal{D}_\mu\Phi^{(n)}|^2 + \frac{1}{g_r^2}|\mathcal{D}_\mu\Phi^{(r)}|^2$$

$$+\left|\mathcal{D}_\mu q^{(1)}\right|^2 + \left|\mathcal{D}_\mu \bar{\tilde{q}}^{(1)}\right|^2 + \left|\mathcal{D}_\mu q^{(2)}\right|^2 + \left|\mathcal{D}_\mu \bar{\tilde{q}}^{(2)}\right|^2 - V_D - V_F, \qquad (2.11)$$

$$V_D = \frac{1}{8}\sum_A \left(\mathrm{Tr}\, t^A \left[\frac{2}{g^2}[\Phi, \Phi^\dagger] + \sum_i (Q_i Q_i^\dagger - \tilde{Q}_i^\dagger \tilde{Q}_i)\right]\right)^2 ;$$

V_F =

$$g_0^2\,|\mu\,\Phi^{(0)} + \sqrt{2}\,\tilde{Q}\,t^{(0)}\,Q|^2 + g_n^2\,|\mu\,\Phi^{(a)} + \sqrt{2}\,\tilde{Q}\,t^{(a)}_{su(n)}\,Q|^2 + g_r^2\,|\mu\,\Phi^{(b)} + \sqrt{2}\,\tilde{Q}\,t^{(b)}_{su(r)}\,Q|^2$$

$$+\tilde{Q}\,[\,M + \sqrt{2}\Phi]\,[\,M + \sqrt{2}\,\Phi]^\dagger\,\tilde{Q}^\dagger + Q^\dagger\,[\,M + \sqrt{2}\,\Phi]^\dagger\,[\,M + \sqrt{2}\,\Phi]\,Q\,, \qquad (2.13)$$

Minimum loop

U_l

$$\begin{pmatrix} e^{i\alpha r}\mathbb{1}_{n\times n} & 0 \\ 0 & e^{i\alpha n}\mathbb{1}_{r\times r} \end{pmatrix}, \qquad \alpha : 0 \to \frac{2\pi}{n\,r},$$

SU_n

$$\begin{pmatrix} e^{i\beta(n-1)/n} & 0 \\ 0 & e^{-i\beta/n}\,\mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix}$$

SU_r

$$\begin{pmatrix} e^{i\gamma(r-1)/r} & 0 \\ 0 & e^{-i\gamma/r}\,\mathbb{1}_{(r-1)\times(r-1)} \end{pmatrix}$$

Global (color-flavor diagonal) symmetry:

$$U(1) \times [SU(n) \times SU(r) \times U(1)]_{C+F}$$

$$\sim \quad U(n) \times U(r)$$

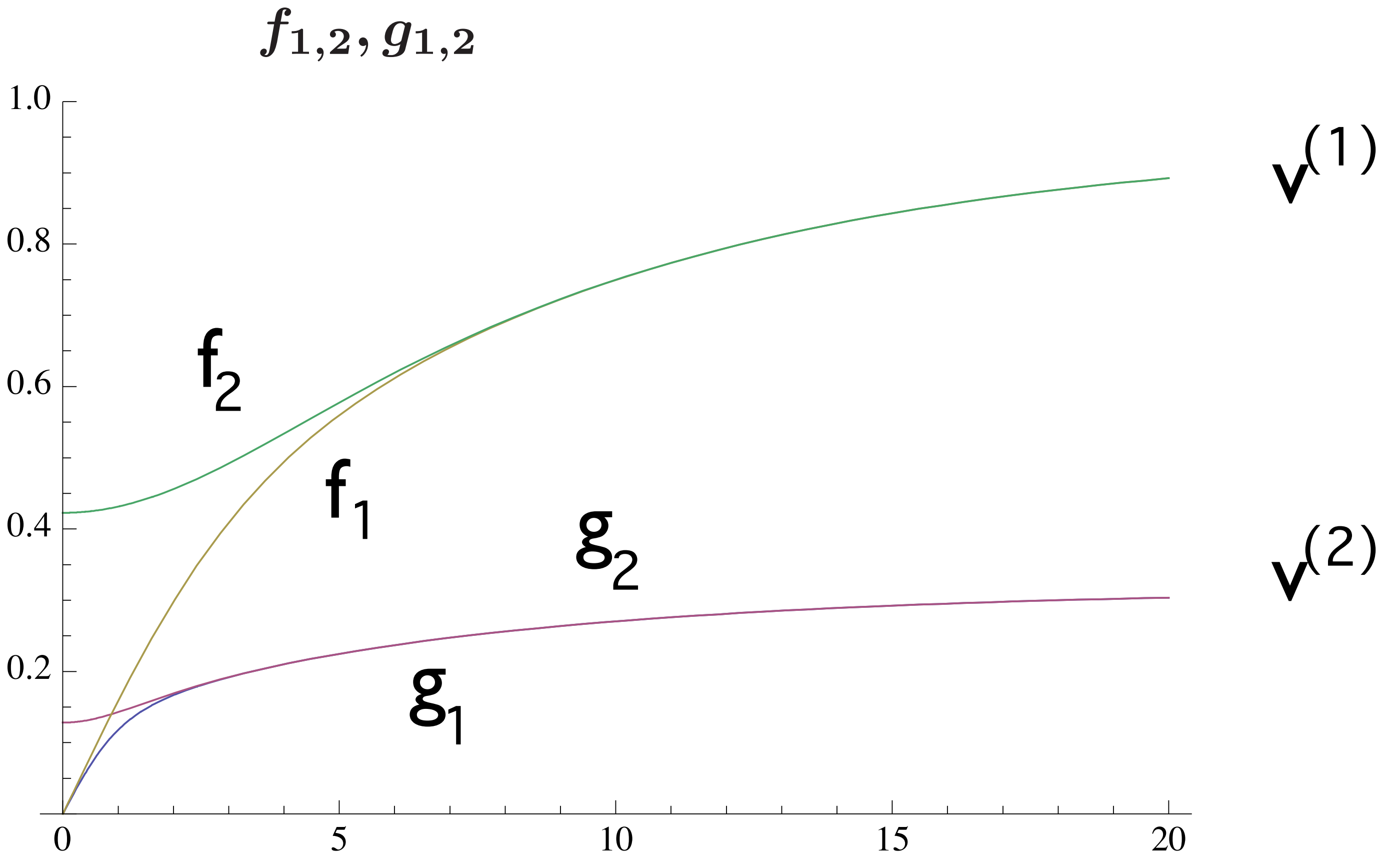
Minimum vortex:

$$\Pi_1 \left(\frac{SU(n) \times SU(r) \times U(1)}{\mathbb{Z}_K} \right) = \mathbf{Z}$$

$$q^{(1)} = \begin{pmatrix} e^{i\phi} f_1(\rho) & 0 \\ 0 & f_2(\rho)\, \mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix}$$

$$\tilde{q}^{(2)} = \begin{pmatrix} e^{i\phi} g_1(\rho) & 0 \\ 0 & g_2(\rho)\, \mathbb{1}_{(r-1)\times(r-1)} \end{pmatrix}$$

$SU(3) \times SU(2) \times U(1)$



Global symmetry "broken" by the vortex:

$$[SU(n) \times SU(r) \times U(1)]_{C+F} \to SU(n-1) \times SU(r-1) \times U(1)^3,$$

Nambu-Goldstone modes propagating only inside the vortex:

⇒ vortex moduli:

$$CP^{n-1} \times CP^{r-1}$$

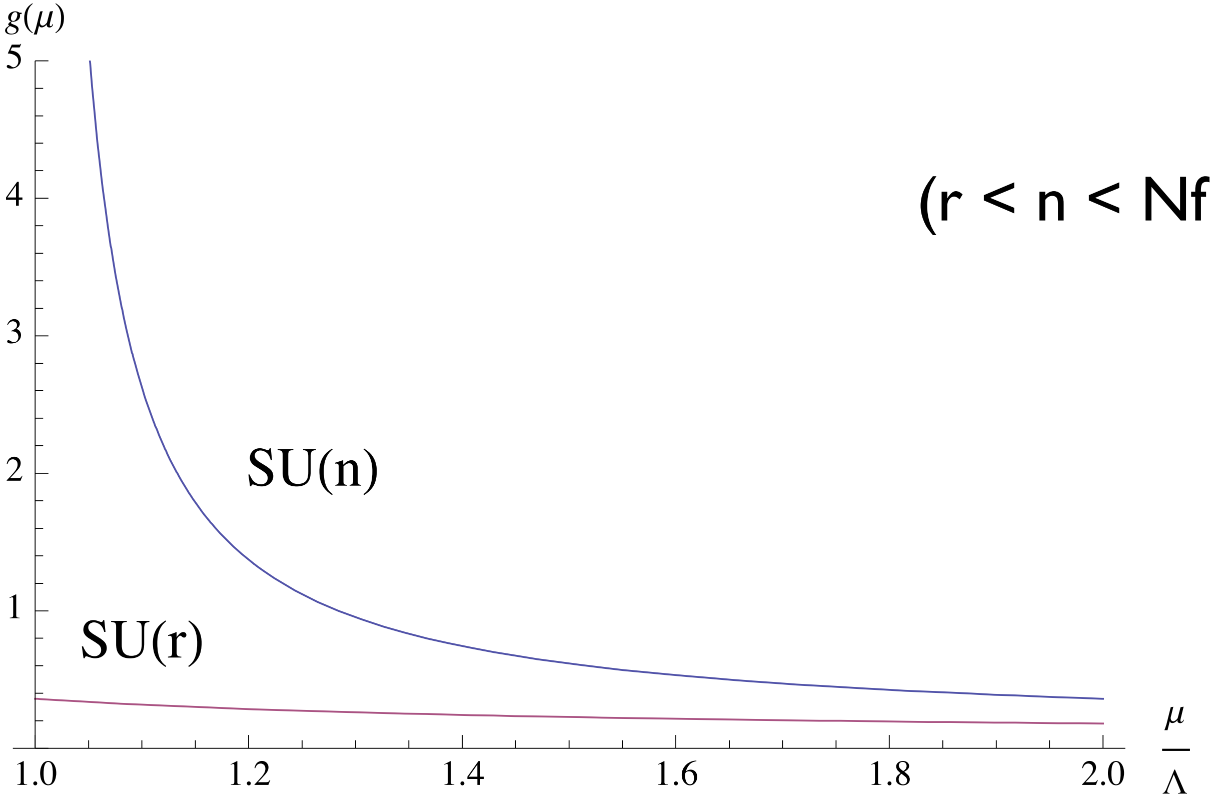
$$\begin{pmatrix} 0 & \mathbb{b}^\dagger & 0 & 0 \\ \mathbb{b} & 0_{(n-1)\times(n-1)} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{c}^\dagger \\ 0 & 0 & \mathbb{c} & 0_{(r-1)\times(r-1)} \end{pmatrix}$$

Vortex orientation can fluctuate along (z, t) ⇒

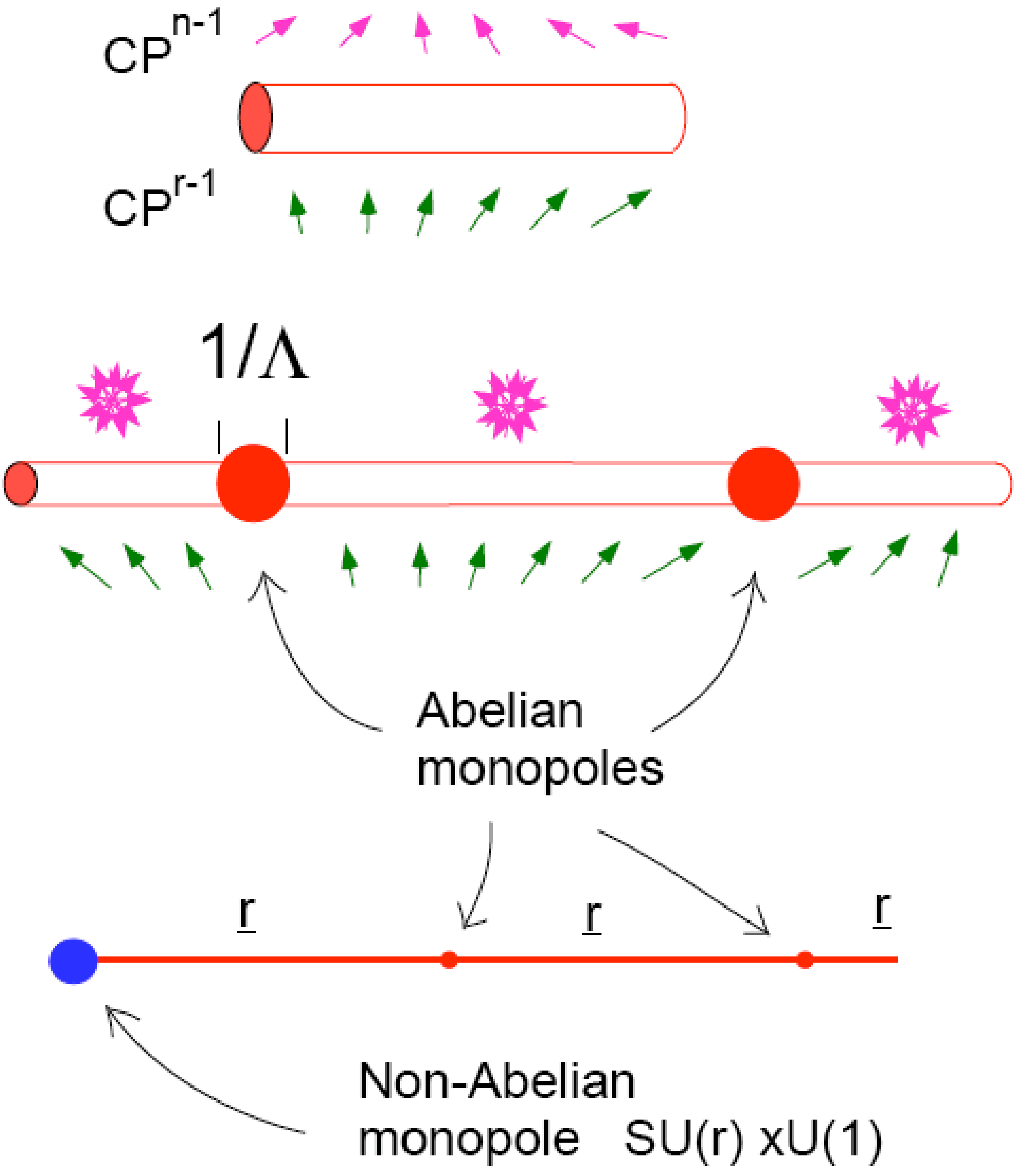
2 D vortex dynamics = $CP^{n-1} \times CP^{r-1}$ sigma model

For n > r, CP^{n-1} interactions become strong and Abelianize, but CP^{r-1} fluctuate still weakly

Monopoles at the end of the vortex carry SU(r) x U(1) quantum number



(r < n < Nf /2)



CP^{n-1}

CP^{r-1}

$1/\Lambda$

Abelian monopoles

$\underline{r}$ $\underline{r}$ $\underline{r}$

Non-Abelian monopole SU(r) xU(1)

4D-2D duality

Vortex dynamics in 2D in 4D (H) theory in Higgs phase

⇔

4D gauge dynamics in G/H theory in Coulomb phase (no H breaking)

Why ?

“Ans : H gauge group restored in the vortex center” ?

No.

$$q = U \begin{pmatrix} e^{i\phi}\,\phi(r) & 0 & \dots & 0 \\ 0 & \chi(r) & 0 & \vdots \\ \vdots & 0 & \chi(r) & 0 \\ 0 & \dots & 0 & \ddots \end{pmatrix} U^\dagger \quad \overset{\rho=0}{\Longrightarrow} \quad U \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & w & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & w \end{pmatrix} U^\dagger$$

Actually, gauge group restored only partially to U(1) in the core

On the other hand, global group smaller inside the vortex

NG modes propagating inside the vortex core

§ 3. Fractional Vortices

('09) M. Eto, M. Nitta, S.B. Gudnason, W. Vinci,
K.K. T. Fujimori, T. Nagashima, K.Ohashi
(Pisa, Tokyo, Cambridge)
B. Collie, D. Tong (Cambridge)
E. Babaev

Def. (here): Vortices with minimum vorticity but with non-trivial tension substructures

(Known examples in EAH; also torons, calorons ....)

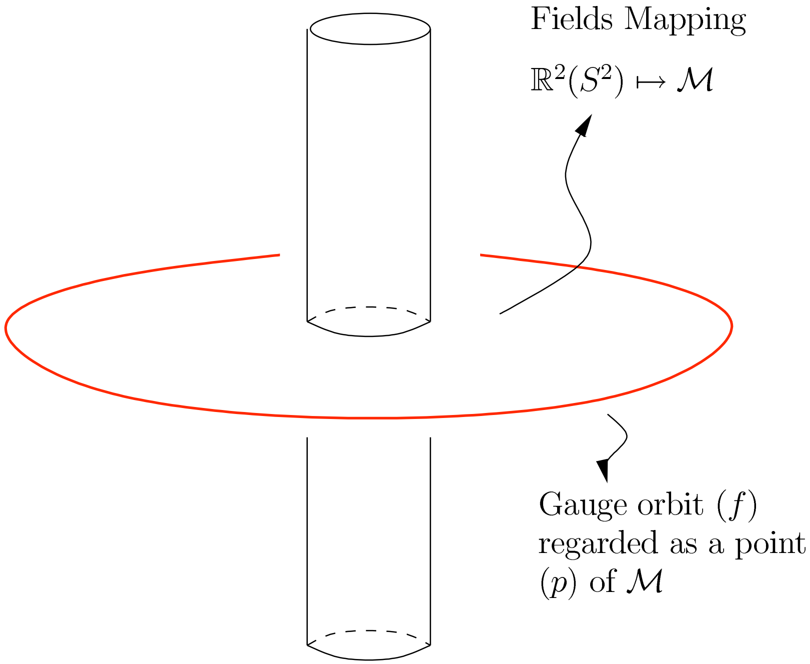
- Various Abelian and non-Abelian generalizations of Abelian Higgs model

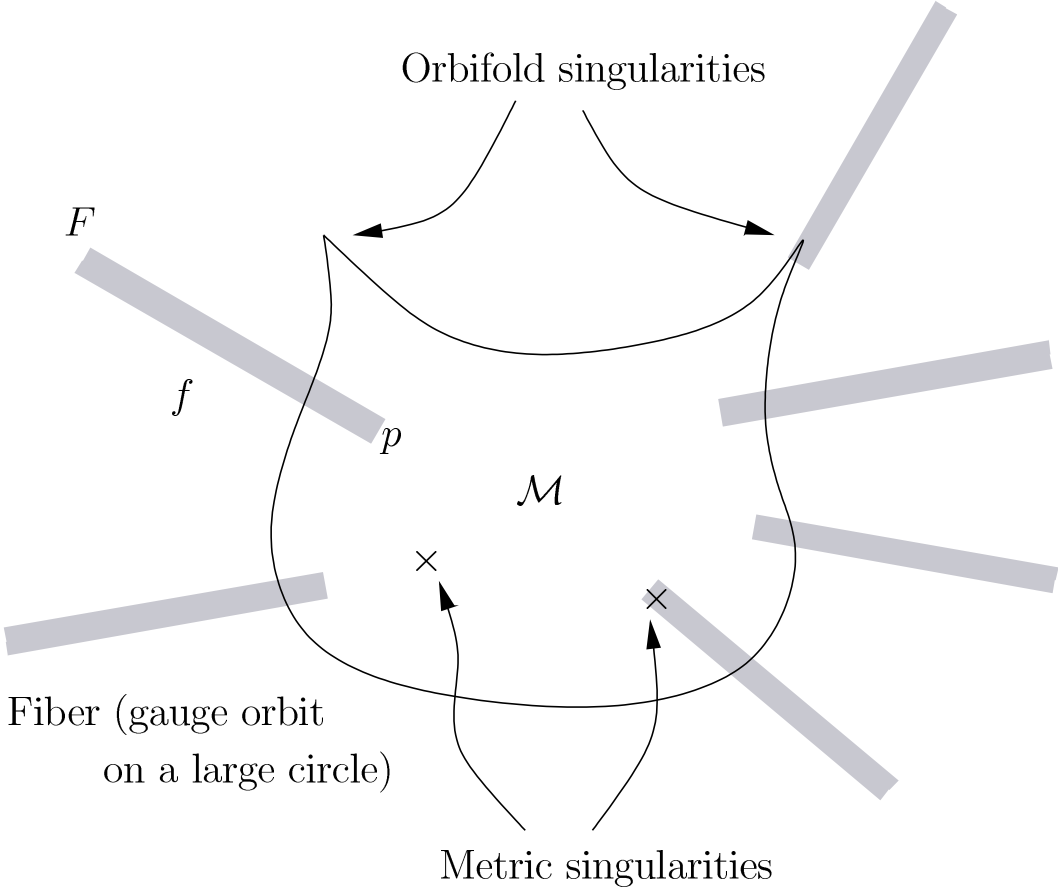
    - BPS (self-dual) nature
    - Vacuum degeneracy ($\mathcal{M}$)

Basic ingredients

- ◆ All vortices $\mathcal{V}$ defined at various points of $\mathcal{M}$ simultaneously
- ◆ $\mathcal{M}$ a singular manifold: ⇒ "fiber bundles over a singular manifold"

- Two distinct mechanisms for fractional peaks


Fields Mapping
$\mathbb{R}^2(S^2) \mapsto \mathcal{M}$
Gauge orbit (f)
regarded as a point
(p) of $\mathcal{M}$


Orbifold singularities
F
f
p
$\mathcal{M}$
Fiber (gauge orbit
on a large circle)
Metric singularities

Exact sequences of fiber bundles

$$\cdots \to \pi_2\left(M, f\right) \to \pi_2\left(\mathcal{M}, p\right) \to \pi_1\left(F, f\right) \to$$

$$\to \pi_1\left(M, f\right) \to \pi_1\left(\mathcal{M}, p\right) \to \cdots$$

$p = \pi(f)$ projection

M= vac. config., F= gauge orbits

f, p = point of F, $\mathscr{M}$

$\mathscr{M}$ = vacuum moduli space = M/F

EAH model : $\Sigma_i |\phi_i|^2 \sim v^2$

$$M = S^{2N-1}\ , \qquad F = S^1\ , \qquad \mathcal{M} = S^{2N-1}/S = \mathbb{C}P^{N-1}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2\left(S^{2N-1}\right) \arrow[r] \arrow[d, equal] & \pi_2\left(\mathbb{C}P^{N-1}\right) \arrow[r] \arrow[d, equal] & \pi_1\left(S^1\right) \arrow[r] \arrow[d, equal] & \pi_1\left(S^{2N-1}\right) \arrow[r] \arrow[d, equal] & \cdots \\
 & \mathbb{1} & \mathbb{Z} & \mathbb{Z} & \mathbb{1} &
\end{tikzcd}$$

The minimum vortex corresponds to a minimum CP^{N-1} lump

Two types of fractional vortices (lumps)

(I) When p =p$_0$ (a Z_N orbifold point) both $\pi_1\left(F, f\right)$ and $\pi_2\left(\mathcal{M}, p\right)$ make a discontinuous change.

Vortex defined near p =p$_0$ feels the presence of p$_0$ and look like a k=N vortex

(II) Even when p is a regular point (not near any singularity), the fields {q} inside S^1 (a disk D^2) ~ $\mathcal{M}$: may hit either one of the singularities or simply pass the region of a large scalar curvature. (Deformed geometry of the sigma model)

cfr. Collie-Tong

Fractional vortex substructures caused either by one of these or by a collaboration of the two ➔ examples

Models based on CP1 (1)

Abelian Higgs model with (A,B), with charges (2,1)

Vacuum config. $2|A|^2 + |B|^2 = \xi \, ,$

Gauge transf: $A \sim e^{2i\alpha(x)} A, \qquad B \sim e^{i\alpha(x)} B.$

➔ $\mathcal{M} = W\mathbb{C}P^1_{2,1} \sim \frac{\mathbb{C}P^1}{\mathbb{Z}_2}$ (Fig.)

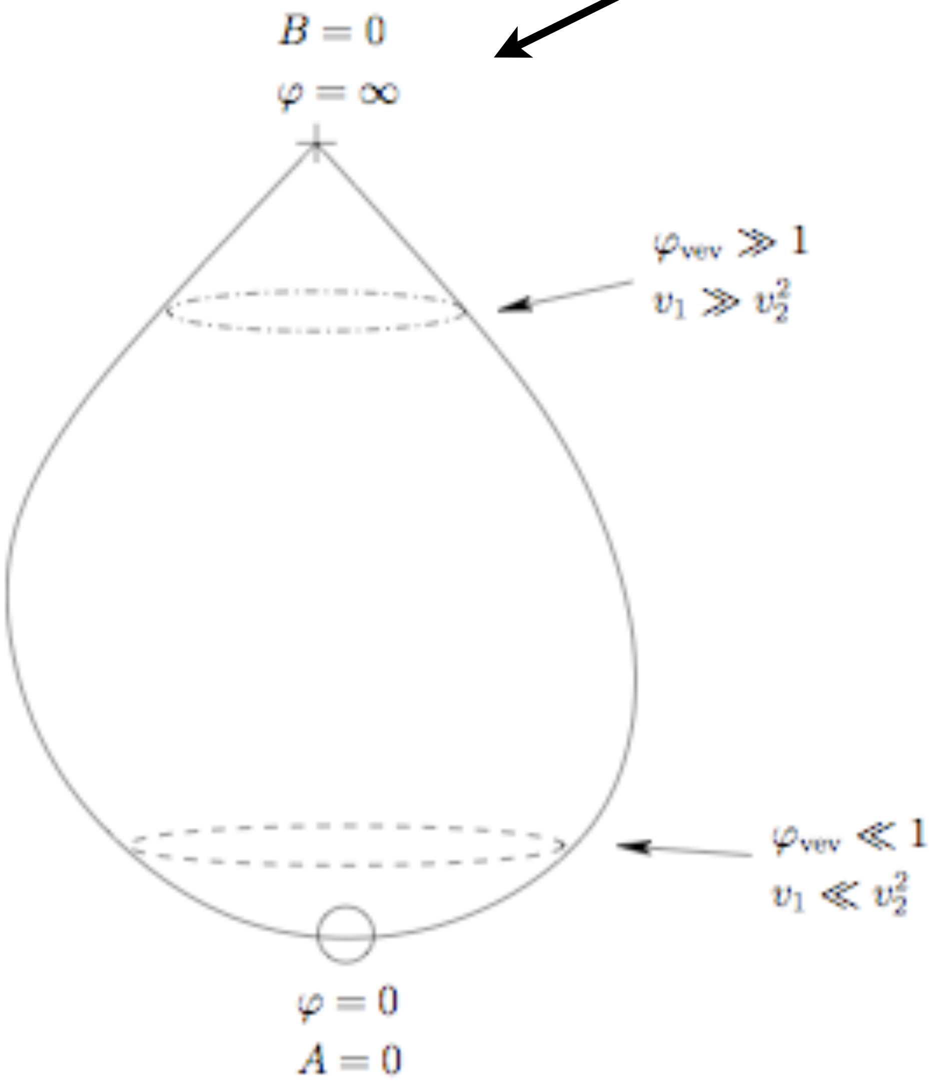
$$H = S^{-1}(z,\bar{z}) H_0(z)$$

The coordinate of $\mathcal{M}$ is $\varphi = 2A/B^2$

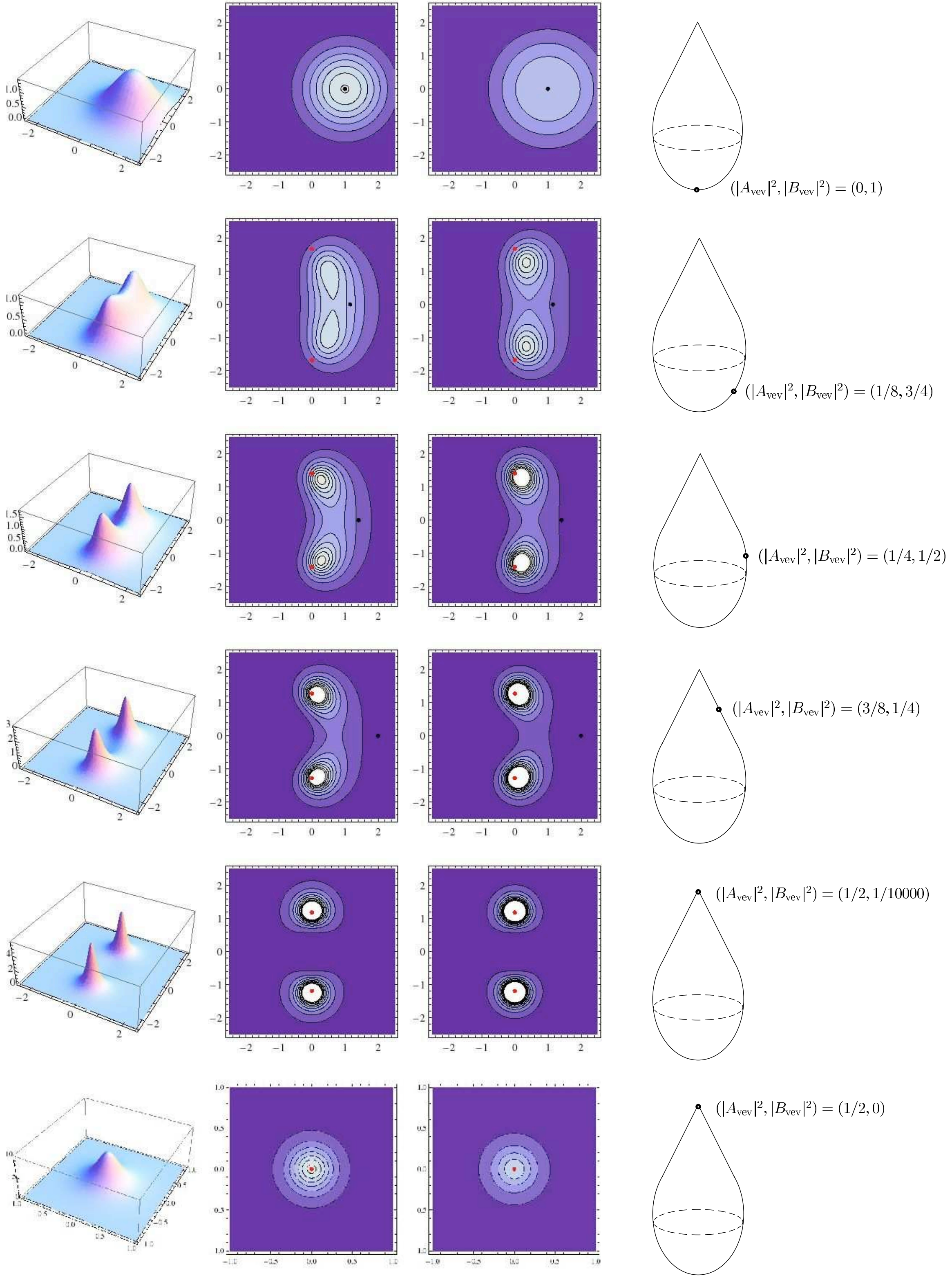
p generic $H_0^{[11]} = \begin{pmatrix} \frac{v_1}{\sqrt{2}}(z-z_1)(z-z_2) & 0 \\ 0 & v_2(z-z_3) \end{pmatrix} \, , \qquad v_1^2 + v_2^2 = \xi$

p= ∞ (B=0) $H_0^{[10]} = \begin{pmatrix} \sqrt{\xi}(z-z_1) & 0 \\ 0 & \zeta_1 \end{pmatrix}$

$$\frac{\pi_2\left(\mathcal{M}, p\right)}{\pi_2\left(\mathcal{M}, \infty\right)} = \mathbb{Z}_2 \, , \qquad \frac{\pi_1\left(F, f\right)}{\pi_1\left(F, f_0\right)} = \mathbb{Z}_2$$



Z_2 orbifold point



black dots = zeros of A;

red dots = zeros of B

Good example of

the first mechanism

Models based on CP^1 (2)

--- U(1)xU(1) Higgs model with (A,B,C) with charges:

$$Q_1 = (2,1,1) \qquad Q_2 = (0,1,-1)$$

$$(A,B,C) \to \left(e^{i2\alpha(x)}A, e^{i\alpha(x)+i\beta(x)}B, e^{i\alpha(x)-i\beta(x)}C\right)$$

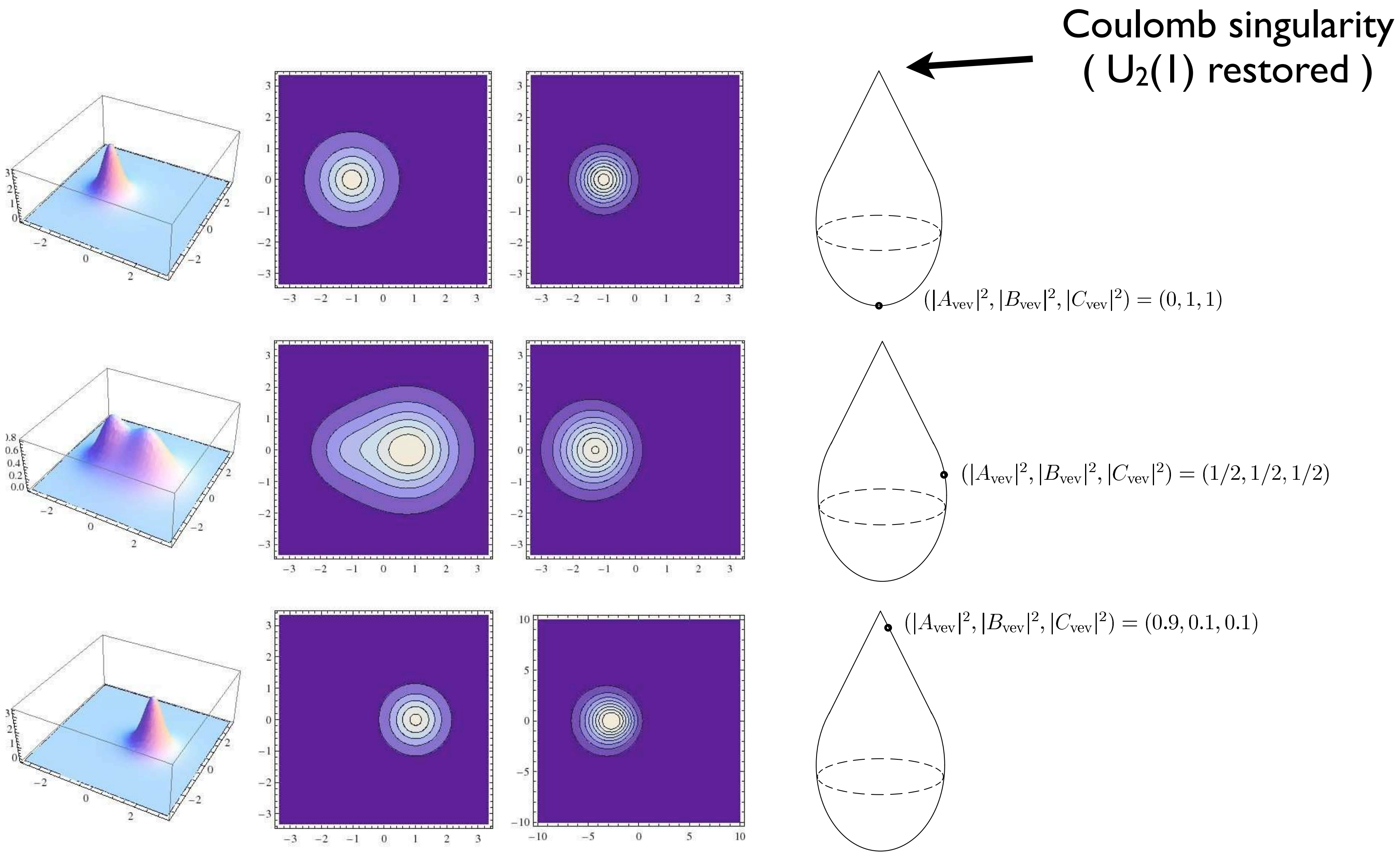
$$U(1)_1 \times U(1)_2/\mathbb{Z}_2.$$

$$\begin{aligned} M &= \{A,B,C \;\;|\;\; 2|A|^2+|B|^2+|C|^2=\xi_1,\; |B|^2-|C|^2=\xi_2\} \\ \mathcal{M} &= M/\left[\left(U(1)_1 \times U(1)_2\right)/\mathbb{Z}_2\right] \;. \end{aligned}$$

No orbifold singularity

No doubling of $\pi_1(F,f)$ or $\pi_2(\mathcal{M},p)$

An extra peak at ~ z=z$_0$, where B(z$_0$) =0



Coulomb singularity
($U_2(1)$ restored)

Fig. 6: The energy density (left-most) and the magnetic flux density $\boldsymbol{F^{(1)}_{12}}$ (2nd from the left), $\boldsymbol{F^{(1)}_{12}}$ (2nd from the right) and the boundary condition (right-most). We have chosen $\boldsymbol{\xi_1 = 2}$, $\boldsymbol{\xi_2 = 0}$, $\boldsymbol{e_1 = 1}$, $\boldsymbol{e_2 = 2}$ and $\boldsymbol{a = -1, b = 1}$ in Eq. (4.34).

Good example of the second mechanism

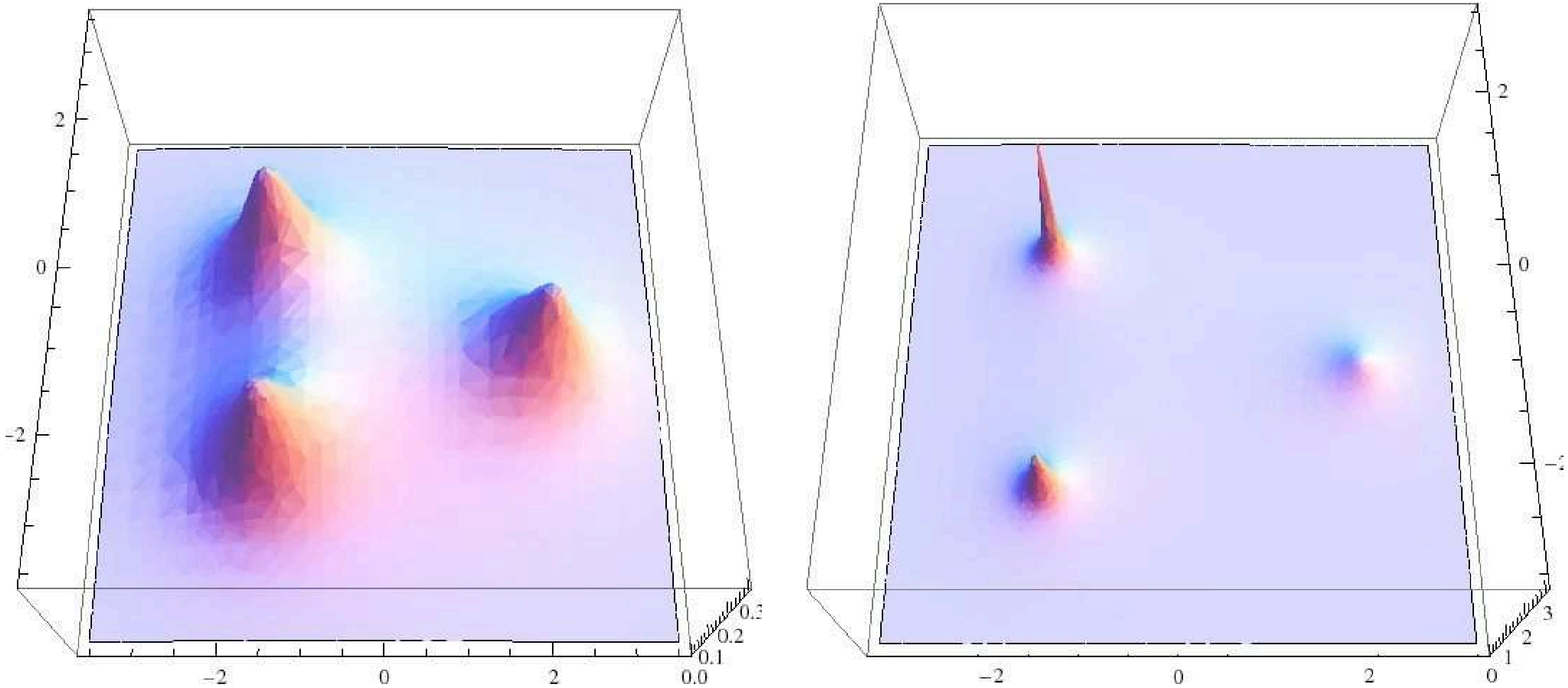
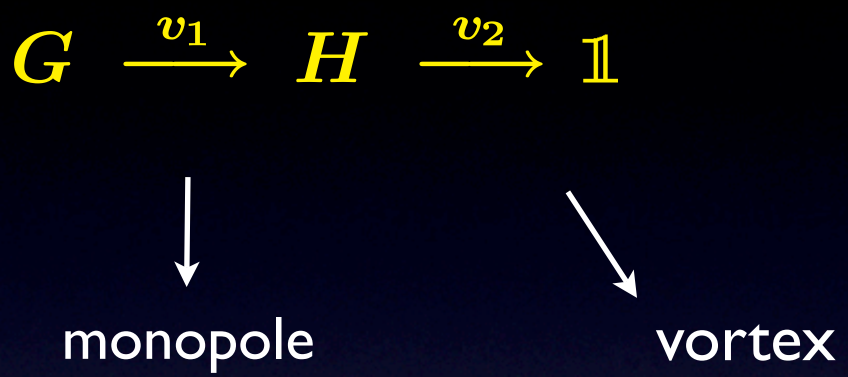


Fig. 11: The energy density of three fractional vortices (lumps) in the $\boldsymbol{U(1) \times SO(6)}$ model in the strong coupling approximation. The positions are $\boldsymbol{z_1 = -\sqrt{2} + i\sqrt{2}, z_2 = -\sqrt{2} - i\sqrt{2}, z_3 = 2}$. *Left panel*: the size parameters are chosen as $\boldsymbol{c_1 = c_2 = c_3 = 1/2}$. *Right panel*: the size parameters are chosen as $\boldsymbol{c_1 = 0, c_2 = 0.1, c_3 = 0.3}$. Notice that one peak is singular ($\boldsymbol{z_1}$) and the other two are regularized by the finite (non-zero) parameters $\boldsymbol{c_{2,3}}$.

§ 4 Monopole - Vortex complex

--- Why the non-Abelian vortices imply
non-Abelian monopoles ---

Hierarchical symmetry breaking



$$G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1} \qquad\qquad v_1 \gg v_2,$$

monopole

vortex

- Apparent paradox (no monopoles, no vortices) ⇨
- Topology and symmetry
  connect monopoles and vortices

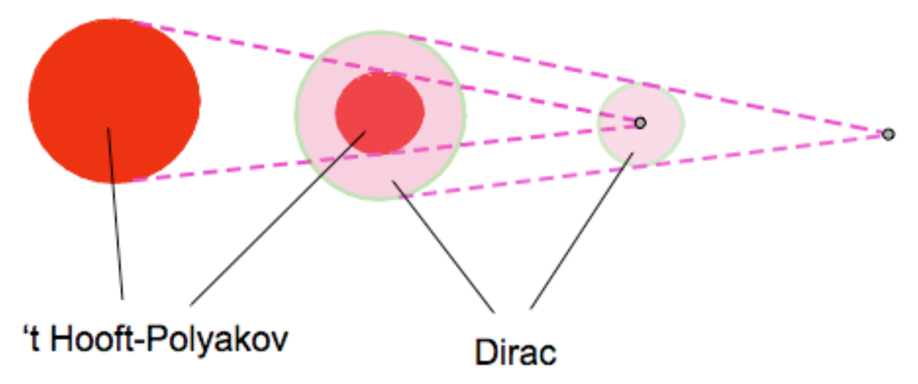
Homotopy-group map

$$G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1} \qquad v_1 \gg v_2,$$

Vortex ! (but also monopole)

Homotopy exact sequence:

$$\cdots \longrightarrow \pi_2(G) \longrightarrow \pi_2(G/H) \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \cdots$$

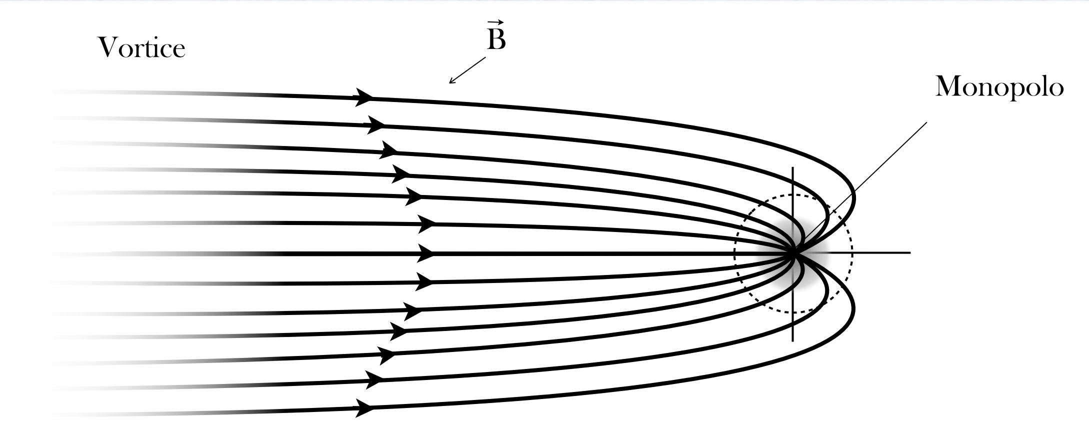


- π_1 (G) = 1 ⇒ Regular monopoles confined by vortices

{
- π_1 (G) = 1 ⇒ All vortices "end" at regular monopoles e.g. SU(N)
- π_1 (G) = Z ⇒ k=2 vortices "end" at regular monopoles! 't Hooft SO(3)/U(1)

cfr., SO(N) k=1 vortices are there: confine Dirac monopoles

Non-Abelian monopole moduli from vortex moduli in the system $G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1}$



Flux matching

(Auzzi-Bolognesi-Evslin-KK; Kneipp)

SU(N+1) ⇒ SU(N) x U(1)

⇒ 1

Exact H_{C+F} induces continuous transformation of vortex --

and monopole

Study in more detail this!

$$SU(3) \to \frac{SU(2) \times U(1)}{Z_2} \to 1.$$

Embedding of 't Hooft-Polyakov soln in S_i = U or V spin

Monopole of the HE theory

$$\phi(\mathrm{r}) = \begin{pmatrix} -\frac{1}{2}v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}v \end{pmatrix} + 3\,v\,\vec{S}\cdot\hat{r}\phi(r),$$

$$\vec{\mathrm{A}}(\mathrm{r}) = \vec{S}\wedge\hat{r}\frac{A(r)}{r}\;;$$

BPS equation

$$B_k^A = -(\mathcal{D}_k\phi)^A$$

Vortex of the LE theory

(BPS)

$$q(x) = \begin{pmatrix} e^{i\,\phi}w_1(\rho) & 0 \\ 0 & w_2(\rho) \end{pmatrix}, \qquad \phi(\mathrm{r}) = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}\;;$$

$$A_i^3(x) = \epsilon_{ij}\,\frac{x_j}{\rho^2}\;(1 - f_3(\rho))\,, \qquad A_\phi^3(\rho) = \frac{1}{\rho}(1 - f_3(\rho)),$$

$$A_i^8(x) = \sqrt{3}\;\varepsilon\epsilon_{ij}\,\frac{x_j}{\rho^2}\;(1 - f_8(\rho))\,, \qquad A_\phi^8(\rho) = \sqrt{3}\frac{1}{\rho}(1 - f_8(\rho)).$$

Interpolating solutions ? Need to take into account a non BPS terms

(Actually both monopole and vortex must be set in the singular gauge)

Ansätze

$$A_\phi = t_3 A^3_\phi(\rho, z) + t_8 A^8_\phi(\rho, z);$$

$$A^3_\phi = -\frac{1}{\rho} f_3(\rho, z), \qquad A^8_\phi = -\sqrt{3}\frac{1}{\rho} f_8(\rho, z),$$

Keep this term

$$\phi(\mathrm{r}) = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix} + \lambda(\rho, z), \qquad \lambda(\rho, z) = t_3\lambda_3(\rho, z) + t_8\lambda_8(\rho, z)$$

$$q(x) = \begin{pmatrix} w_1(\rho, z) & 0 \\ 0 & w_2(\rho, z) \end{pmatrix} .$$

Coupled (quadratic) equations for 6 profile functions which reduce to the linear BPS equations for λ =0

$\Rightarrow$

- The Dirac string of the monopole hidden deep in the vortex core

- The whole monopole-vortex complex breaks $SU(2)_{C+F}$:
  orientational zeromodes of $SU(2)/U(1) \sim CP^1$

- The degeneracy between the monopole solution living in (13) SU(2) subgroup and that in (23) SU(2) subgroup is explicitly broken by the vortex -- failure of the naïve "non-Abelian monopole" concept (multiplet of H)

- An exact $SU(2)_{C+F}$ symmetry ⇒ Degeneracy (and indeed continous CP^1 degeneracy)

under the simultaneous color-flavor rotations for the monopole -- vortex complex

- It is a magnetic symmetry, i.e., symmetry of magnetic-flux orientation

- A new exact symmetry for the monopole: the origin of the dual SU(2) group (multiplet of $\tilde{H}$)

Cipriani,Dorigoni, Gudnason, Konishi, Michelini,
in preparation

Conclusion

- Non-Abelian vortices and generalizations -- a vast variety of phenomena implied by such solutions: true reach of these equations and solutions yet to be seen

- Many intriguing results encompassing physics of strong gauge dynamics, confinement and symmetry breaking, and perhaps, interesting mathematics

- Non-Abelian monopoles (GNO duality) from the monopole-vortex complex

$$\tilde{H} \sim H_{C+F}$$

- (Dual) confinement mechanism of non-Abelian variety

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