

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...
m_1	m_2	
m_1	m_2	Coefficients
.	.	
.	.	

$1/2 \times 1/2$

1
+1/2 +1/2
+1/2 -1/2
-1/2 +1/2
-1/2 -1/2

 $Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$
 $2 \times 1/2$

5/2
+5/2
+2 +1/2
+2 -1/2
+1 +1/2

 $Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$
 $1 \times 1/2$

3/2
+3/2
+1 +1/2
+1 -1/2
0 +1/2

 $Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$
 2×1

3
+3
+2 +1
+2 0
+1 +1

 $Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$
 $3/2 \times 1/2$

2
+2
+3/2 +1/2
+3/2 -1/2
+1/2 +1/2

 $Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$
 1×1

2
+2
+1 +1
+1 0
0 +1

 $3/2 \times 1$

5/2
+5/2
+3/2 +1
+3/2 0
+1/2 +1

 $3/2 \times 3/2$

3
+3
+3/2 +3/2
+3/2 +1/2
+1/2 +3/2

 $2 \times 3/2$

7/2
+7/2
+2 +3/2
+2 +1/2
+1 +3/2

 2×2

4
+4
+2 +2
+2 +1
+1 +2

 $Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$
 $d_{m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$
 $\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle$
 $= (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M \rangle$
 $d_{m',m}^j = (-1)^{m-m'} d_{m,m'}^j = d_{-m,-m'}^j$
 $d_{0,0}^1 = \cos \theta$
 $d_{1/2,1/2}^{1/2} = \cos \frac{\theta}{2}$
 $d_{1,1}^1 = \frac{1 + \cos \theta}{2}$
 $d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2}$
 $d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$
 $d_{1,-1}^1 = \frac{1 - \cos \theta}{2}$
 $d_{3/2,3/2}^{3/2} = \frac{1 + \cos \theta}{2} \cos \frac{\theta}{2}$
 $d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1 + \cos \theta}{2} \sin \frac{\theta}{2}$
 $d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1 - \cos \theta}{2} \cos \frac{\theta}{2}$
 $d_{3/2,-3/2}^{3/2} = -\frac{1 - \cos \theta}{2} \sin \frac{\theta}{2}$
 $d_{1/2,1/2}^{3/2} = \frac{3 \cos \theta - 1}{2} \cos \frac{\theta}{2}$
 $d_{1/2,-1/2}^{3/2} = -\frac{3 \cos \theta + 1}{2} \sin \frac{\theta}{2}$
 $d_{2,2}^2 = \left(\frac{1 + \cos \theta}{2} \right)^2$
 $d_{2,1}^2 = -\frac{1 + \cos \theta}{2} \sin \theta$
 $d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta$
 $d_{2,-1}^2 = -\frac{1 - \cos \theta}{2} \sin \theta$
 $d_{2,-2}^2 = \left(\frac{1 - \cos \theta}{2} \right)^2$
 $d_{1,1}^2 = \frac{1 + \cos \theta}{2} (2 \cos \theta - 1)$
 $d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta$
 $d_{1,-1}^2 = \frac{1 - \cos \theta}{2} (2 \cos \theta + 1)$
 $d_{0,0}^2 = \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.