Buco finito.

Int. $\psi'' = -k^2\psi \Rightarrow \sin kx\,,\ \cos kx$

Est. $\psi'' = \kappa^2\psi\,,\quad \kappa^2 = \frac{2m(V_0-E)}{\hbar^2}$

$$\psi = e^{\pm\kappa x} \rightarrow \psi = e^{-\kappa x} \quad x > a/2$$
$$= e^{\kappa x} \quad x < -a/2.$$

Sol. pari

$$\psi^{i} = A\cos kx \qquad \psi^{est} = B e^{-\kappa x} \quad x > a/2$$
$$= B e^{\kappa x} \quad x < -\frac{a}{2}$$

Cond. raccordo

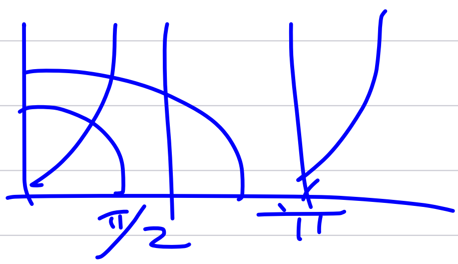
$x = a/2$, ψ: $\quad A\cos\frac{ka}{2} = B e^{-\kappa a/2} \qquad (1)$

ψ': $\quad -Ak\sin\frac{ka}{2} = -B\kappa e^{-\kappa a/2} \qquad (2)$

$\cancel{(1)}/(2) \Rightarrow -k\tan\frac{ka}{2} = -\kappa \qquad \xi^2+\eta^2 = \frac{a^2}{4}(k^2+\kappa^2)$

$\frac{ka}{2} = \xi\ ;\ \frac{\kappa a}{2} = \eta \Rightarrow \qquad = \frac{mVa^2}{2\hbar^2}$

$$\begin{cases} \eta = \xi\tan\xi \\ \xi^2+\eta^2 = \frac{mV_0a^2}{2\hbar^2} \end{cases}$$



Sol. dispari.

$$\psi^{int} = A \sin kx \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi^{est} = B e^{-Kx} \quad (x > a/2) \qquad K = \frac{\sqrt{2m(V-E)}}{\hbar}$$

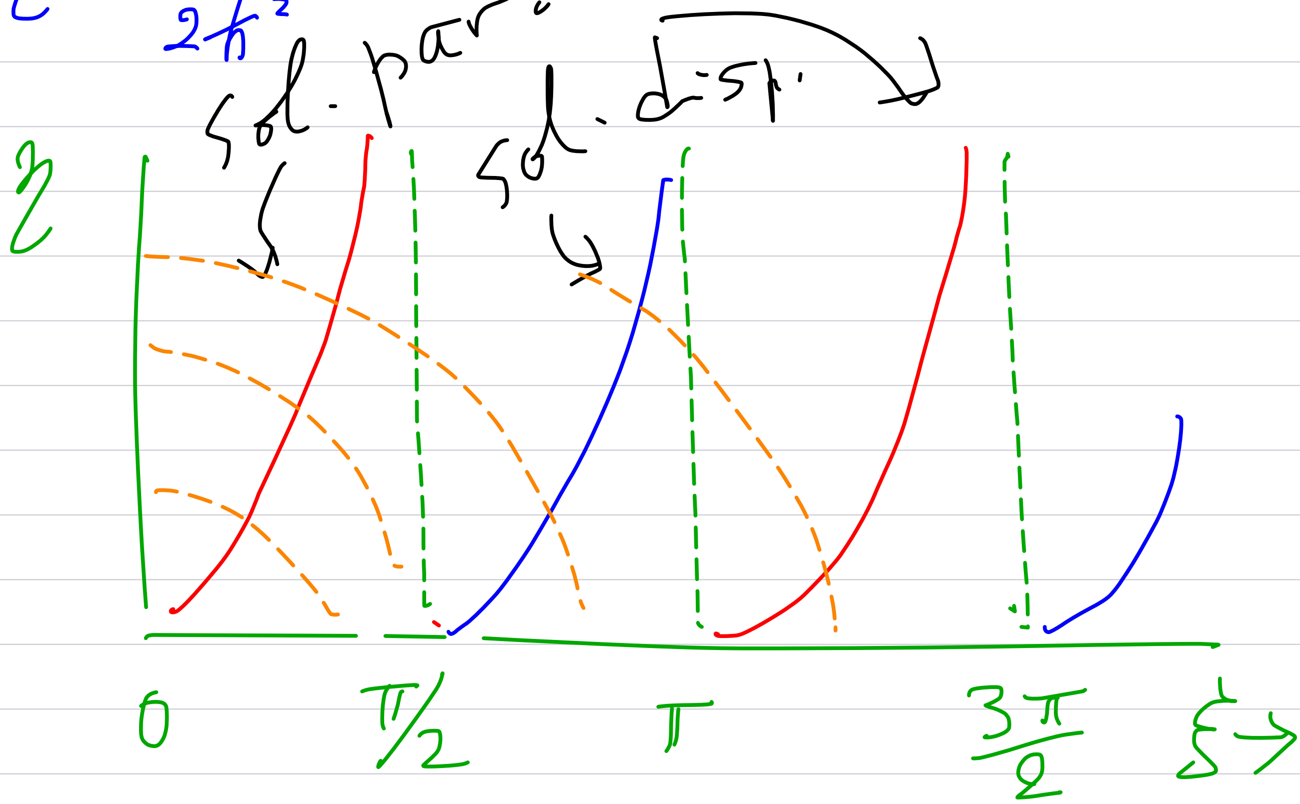
$$= -B e^{Kx} \quad (x < -a/2)$$

Cond. raccordo a $x = a/2$

$$\begin{cases} A \sin \frac{ka}{2} = B e^{-Ka/2} \\ k A \cos \frac{ka}{2} = -K \cdot B e^{-Ka/2} \end{cases}$$

$$\Rightarrow \quad -\xi \cot \xi = \eta, \quad \xi = \frac{ka}{2}, \; \eta = \frac{Ka}{2}.$$

$$\begin{cases} \xi^2 + \eta^2 = \frac{mVa^2}{2\hbar^2} \end{cases}$$



(i) $0 < \sqrt{\frac{mVa^2}{2\hbar^2}} \leq \pi/2 \Rightarrow$ 1 stato legato

(ii) $\pi/2 < \sqrt{} \leq \pi \Rightarrow$ 2 stati legati

$\ldots$

(iii) $\frac{\pi n}{2} < \sqrt{} \leq \frac{\pi(n+1)}{2} \Rightarrow n+1$ stati legati

$\sim$

Casi critici:

$$\sqrt{\frac{mVa^2}{2\hbar^2}} = \frac{\pi(n+1)}{2}, \quad n = 0, 1, 2, \ldots$$

il nuovo stato legato?

ma $K = 0$!

$$\psi^{est.} = e^{-Kx} = \text{cost}, \quad x > 0$$

NON è ancora uno stato legato:

È uno stato del continuo che sta per diventare uno stato legato!!

Limiti vari.

(i) $V_0 \to \infty$, a fisso

$$\Rightarrow \xi = \frac{\pi n}{2}, \quad k_n = \frac{\pi n}{a}, \quad \text{ok.}$$

(ii) $V_0 \to 0$, e/o, $a \to 0$ (ma non nullo)

$$\sqrt{\frac{mV_0 a^2}{2\hbar^2}} \ll \pi/2 \Rightarrow$$ 1 solo stato legato.

(iii) $V_0 \to \infty$, $a \to 0$, $V_0 a = f$, fisso

$$\sqrt{\frac{mV_0 a^2}{2\hbar^2}} = \sqrt{\frac{mf}{2\hbar^2}}\sqrt{a} \to 0$$ $\therefore$ 1 solo stato legato.

$$\left.\begin{aligned} E - V_0 &\equiv \tilde{E} \\ V(x) - V_0 &\equiv \tilde{V}(x) \end{aligned}\right\} \to \tilde{V}(x) = -f\,\delta(x)$$

$$-\frac{\hbar^2}{2m}\psi'' + \tilde{V}\psi = \tilde{E}\psi \; !$$

Sol. d.

$$\left\{ \begin{array}{l} \eta = \xi \tan \xi \\ \xi^2 + \eta^2 = \dfrac{m V_0 a^2}{2\hbar^2} \end{array} \right\} \text{nel limite } V \to 0 .$$

$$\Rightarrow \xi, \eta \ll 1 \Longrightarrow \eta \sim \xi^2 \ll \xi$$

$$\xi \sim \sqrt{\frac{m V_0 a^2}{2\hbar^2}} \Longrightarrow$$

$$\eta \approx \xi^2 \sim \frac{m V_0 a^2}{2\hbar^2} = \frac{\kappa a}{2} = \frac{\sqrt{2m(V-E)}}{2\hbar} a$$

$$\therefore \quad V_0 - E = -\tilde{E}$$

$$= \frac{1}{2m}\left(\frac{m V_0 a}{\hbar}\right)^2 = \frac{m f^2}{2\hbar^2} \quad !$$

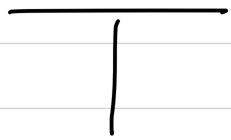
$$\tilde{E} = -m f^2 / 2\hbar^2$$

$$\eta = \frac{\kappa a}{2} \quad \therefore \quad \kappa = \frac{2}{a}\eta = \frac{\cancel{2}}{a}\frac{m V_0 a^2}{\cancel{2}\hbar^2}$$

$$= \frac{m f}{\hbar^2} \quad \text{(finito!)}$$

Stato legato in una buca $\delta(x)$

$V(x) = -f \cdot \delta(x)$



Condiz. di Discontinuità attraverso $\delta(x)$.

$$-\frac{\hbar^2}{2m}\psi'' - f\delta(x)\psi = E\psi(x)$$

$$\int_{-\epsilon}^{\epsilon} dx \Rightarrow -\frac{\hbar^2}{2m}\left(\psi'_+ - \psi'_-\right) - f\psi(0) = O(\epsilon)$$

$$\therefore\ \psi'_+ - \psi'_- = -\frac{2mf}{\hbar^2}\psi(0) \qquad (*)$$

Per risolvere l'eq. Sch. (per $E<0$)

$$\psi(x) = A\,e^{-\kappa x}, \quad x>0 \quad \left(\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}\right)$$

$$= A\,e^{\kappa x} \qquad x<0.$$

($\psi_+(0) = \psi_-(0)$ è stato imposto)

(La cond. Normalizzabilità $\checkmark$)

Raccordo $(*)$ $\Rightarrow$

$$-2\kappa A = -\frac{2mf}{\hbar^2}A$$

ma

$$\therefore\ \boxed{\kappa = \frac{mf}{\hbar^2}} \quad \text{OK.} \quad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

$$E = -\frac{\kappa^2\hbar^2}{2m} = -\left(\frac{mf}{\hbar^2}\right)^2\frac{\hbar^2}{2m}$$
$$= -\frac{mf^2}{2\hbar^2} \qquad \text{ok.} \checkmark$$

Osservazione.

(i) È l'unico stato legato.

(ii) $E \geq 0 \rightarrow$ stati di continuo.

(Soluz. problemi di diffusione)

Formulazione del problema.

- Trovare le soluz. generale $E \geq 0$.
- Imporre le condiz. appropriate.

e.g. particella inc. da $x = -\infty$;

e.g. " inc. da $x = +\infty$.

e.g. particella legata al potenziale "ionizza" e decade.

Quindi troviamo le sol. generali

Lo spettro continuo

$$-\frac{\hbar^2}{2m}\psi'' - f\,\delta(x)\,\psi = E\psi \quad \left(\begin{array}{l} f>0 \text{ attrattivo} \\ f<0 \text{ ripulsivo} \end{array} \right.$$

Sol. $\psi = A e^{ikx} + B e^{-ikx} \qquad x<0$

$\quad = C e^{ikx} + D e^{-ikx} \qquad x>0$.

$$\begin{cases} A+B = C+D \,. & (*) \\ ik(C-D) - ik(A-B) = -\frac{2mf}{\hbar^2}(C+D) \,. \end{cases}$$

$\hookrightarrow \left(1+\frac{2mf}{ik\hbar^2}\right)C - \left(1-\frac{2mf}{ik\hbar^2}\right)D = A-B \quad (**)$

$(*)\,(**) \Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1-i\alpha & i\alpha \\ -i\alpha & 1+i\alpha \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \qquad \alpha = \alpha(k)$

Questa rappr. la soluz. completa.

(Doppia degenerazione $k \neq 0$)

$\hookrightarrow C=0$ o $D=0$ per es.

per $D=0$ es. $\quad A = (1-i\alpha)C$

$\qquad B = -i\alpha C$

Problema di diffusione.

e.g. Particella incidente da $x = -\infty$

$--\to$

Cond. al contorno fisico è:

$D = 0$.

Per il prob. della diffusione, le quantità di interesse sono:

$$j^{inc} = |A|^2 \frac{\hbar k}{m} \to v$$

$$\left.\begin{aligned} j^{rif} &= |B|^2 \hbar k/m, \\ j^{trasm} &= |C|^2 \hbar k/m. \end{aligned}\right\} \Rightarrow$$

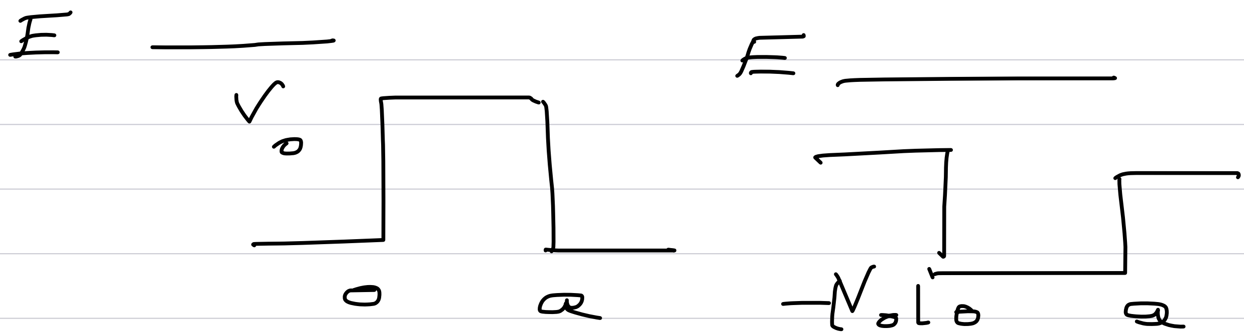
D = Coeff. di trasmissione

$$= |C|^2/|A|^2 = \frac{1}{1+\alpha^2}$$

R = Coeff di riflessione

$$= |B|^2/|A|^2 = \frac{\alpha^2}{1+\alpha^2}.$$

Diffusione dalla $\left\{\begin{array}{l}\text{buca}\\ \text{barriera}\end{array}\right\}$ di potenziale.



$$V(x) = 0 \qquad x<0 \; ; \; x>a$$
$$\left\{ \quad = V_0 \qquad 0<x<a. \right.$$

$E > V_0$.

$k = \frac{\sqrt{2mE}}{\hbar}$; $k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

Sol. est.

$$\psi = Ae^{ikx} + Be^{-ikx} \qquad x<0$$
$$= Ce^{ik'x} + De^{-ik'x} \qquad 0<x<a.$$
$$= Ee^{ikx} + Fe^{-ikx} \qquad x>a.$$

- Risolvere $\begin{pmatrix} E \\ F \end{pmatrix}$ in termini di $\begin{pmatrix} A \\ B \end{pmatrix}$

o vice versa.

- Cond. raccordo a $x=0$; $x=a$
- Eliminare C, D.

Cond.

$x=0$ $\begin{cases} A+B = C+D \\ ik(A-B) = ik'(C-D) \end{cases}$

$$A-B = \frac{k'}{k}(C-D)$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix} 1+\frac{k'}{k} & 1-\frac{k'}{k} \\ 1-\frac{k'}{k} & 1+\frac{k'}{k} \end{pmatrix}}_{M\left(\frac{k'}{k}\right)}\begin{pmatrix} C \\ D \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \left(k \leftrightarrow k' \right)\begin{pmatrix} A \\ B \end{pmatrix}.$$

Cond

$x=a$.

$$e^{ik'a}C + e^{-ik'a}D = e^{ika}E + e^{-ika}F$$

$$ik'\left(e^{ik'a}C - e^{-ik'a}D\right) = ik\left(e^{ika}E - e^{-ika}F\right)$$

Set $\begin{cases} e^{ik'a}C \equiv \tilde{C}\ ; & e^{-ik'a}D = \tilde{D} \\ e^{ika}E = \tilde{E}\ ; & e^{-ika}F = \tilde{F} \end{cases}$

allora

$$\begin{pmatrix} \tilde{C} \\ \tilde{D} \end{pmatrix} = M\left(\frac{k}{k'}\right)\begin{pmatrix} \tilde{E} \\ \tilde{F} \end{pmatrix}.$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} e^{-ik'a} & 0 \\ 0 & e^{ik'a} \end{pmatrix}\begin{pmatrix} \tilde{C} \\ \tilde{D} \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \underbrace{M\left(\frac{k'}{k}\right)\begin{pmatrix} e^{-ik'a} & \\ & e^{ik'a} \end{pmatrix} M\left(\frac{k}{k'}\right)}_{N} \begin{pmatrix} \tilde{E} \\ \tilde{F} \end{pmatrix} \quad !$$

$$N = \frac{1}{2}\begin{pmatrix} 1+\frac{k'}{k} & 1-\frac{k'}{k} \\ 1-\frac{k'}{k} & 1+\frac{k'}{k} \end{pmatrix}\begin{pmatrix} e^{-ik'a} & \\ & e^{ik'a} \end{pmatrix}\begin{pmatrix} 1+\frac{k}{k'} & 1-\frac{k}{k'} \\ 1-\frac{k}{k'} & 1+\frac{k}{k'} \end{pmatrix}\frac{1}{2}.$$

$$= \frac{1}{4}\begin{pmatrix} 1+\frac{k'}{k} & 1-\frac{k'}{k} \\ 1-\frac{k'}{k} & 1+\frac{k'}{k} \end{pmatrix}\begin{pmatrix} e^{-ik'a}\left(1+\frac{k}{k'}\right) & e^{-ik'a}\left(1-\frac{k}{k'}\right) \\ e^{ik'a}\left(1-\frac{k}{k'}\right) & e^{ik'a}\left(1+\frac{k}{k'}\right) \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} e^{-ik'a}\left(1+\frac{k'}{k}\right)\left(1+\frac{k}{k'}\right)+e^{ik'a}\left(1-\frac{k'}{k}\right)\left(1-\frac{k}{k'}\right) & e^{-ik'a}\left(1+\frac{k'}{k}\right)\left(1-\frac{k}{k'}\right)+e^{ik'a}\left(1-\frac{k'}{k}\right)\left(1+\frac{k}{k'}\right) \\ - \quad + \qquad + \quad - & - \quad - \qquad + \quad + \end{pmatrix}$$

$$= \begin{pmatrix} \cos k'a - \frac{i}{2}\left(\frac{k'}{k}+\frac{k}{k'}\right)\sin k'a. & -\frac{i}{2}\left(\frac{k'}{k}-\frac{k}{k'}\right)\sin k'a. \\ \frac{i}{2}\left(\frac{k'}{k}-\frac{k}{k'}\right)\sin k'a & \cos k'a + \frac{i}{2}\left(\frac{k'}{k}+\frac{k}{k'}\right)\sin k'a \end{pmatrix}$$

$$\Rightarrow$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = N \begin{pmatrix} \tilde{E} \\ \tilde{F} \end{pmatrix}$$

Problema della diffusione.

Part. incidente da $x = -\infty$

∴ $\psi = A e^{ikx} + B e^{-ikx}$ $\quad x < 0$

$= E e^{ikx}$ $\quad x > a$.

∴ $F = 0$ ∴ $\tilde{F} = 0$.

$j_{inc} = |A|^2 \frac{k\hbar}{m}$;

$j_{rif} = |B|^2 \left(-\frac{k\hbar}{m}\right)$.

$j_{trasm.} = |E|^2 \frac{k\hbar}{m}$.

∴ $D = \text{Prob. Trasm} = \frac{|E|^2}{|A|^2}$ (Coeff. di trasm.)

$R = \text{Prob riflessione} = \frac{|B|^2}{|A|^2}$ (Coeff di Rifless.)

∴ Calcolare $\left|\frac{B}{A}\right|^2$, $\left|\frac{E}{A}\right|^2 = \left|\frac{\tilde{E}}{A}\right|^2$.

Set $A = 1$.

$$\begin{pmatrix} 1 \\ B \end{pmatrix} = \begin{pmatrix} \cos k'a - \frac{i}{2}\left(\frac{k'}{k}+\frac{k}{k'}\right)\sin k'a & -\frac{i}{2}\left(\frac{k'}{k}-\frac{k}{k'}\right)\sin k'a \\ \frac{i}{2}\left(\frac{k'}{k}-\frac{k}{k'}\right)\sin k'a & \cos k'a + \frac{i}{2}\left(\frac{k'}{k}+\frac{k}{k'}\right)\sin k'a \end{pmatrix} \begin{pmatrix} \tilde{E} \\ 0 \end{pmatrix}$$

$$\therefore \quad \tilde{E} = \frac{1}{\cos k'a - \frac{i}{2}\left(\frac{k'}{k}+\frac{k}{k'}\right)\sin k'a} \qquad (*)$$

$$B = \frac{\frac{i}{2}\left(\frac{k'}{k}-\frac{k}{k'}\right)\sin k'a}{\cos k'a - \frac{i}{2}\left(\frac{k'}{k}+\frac{k}{k'}\right)\sin k'a} \qquad (**)$$

$$\therefore \quad D = |\tilde{E}|^2 = \frac{|2kk'|^2}{|2kk'\cos k'a - i(k^2+k'^2)\sin k'a|^2}$$

$$= \frac{4k^2k'^2}{4k^2k'^2(1-\sin^2 k'a) + (k^2+k'^2)^2\sin^2 k'a}.$$

$$= \frac{4k^2k'^2}{4k^2k'^2 + (k^2-k'^2)^2\sin^2 k'a}$$

$$R = |B|^2 = \frac{(k^2-k'^2)^2\sin^2 k'a}{4k^2k'^2 + (k^2-k'^2)^2\sin^2 k'a}$$

$D+R=1$

Osservazione.

(i) $R \neq 0$ anche se $E > V_0$ (cfr. M. Cl.)

(ii) Ma per certi valori di k, i.e. tale che

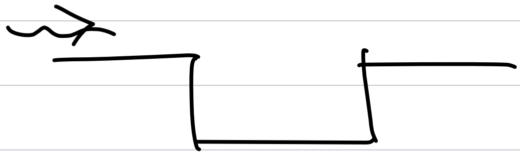
$$k'a = \pi n, \quad (n = 1, 2, 3 \dots)$$

$D = 1$ (trasmissione totale)!

$R = 0$.

Nota come effetto Ramsauer-Taunsend (Diffusione da atomi; in 3D solo per potenziali attrattivi.)

(iii) $k \to 0$ $\quad V_0 < 0$



$$\left.\begin{array}{l} D = 0 \\ R = 1 \end{array}\right\} \text{Riflessione totale.}$$

(cfr. M. Cl.).

(iv) $k' \to 0$ $\quad V_0 > 0$

$D = 0$

$R = 1$

(v) $V_0 < 0 \quad |V_0| \to \infty$.

$a \to 0, \quad V_0 a = -f$ (fisso)

$$D = \frac{4k^2 k'^2}{4k^2k'^2 + (k^2 - k'^2)^2 \sin^2 k'a}$$

$$k' = \frac{\sqrt{2m(E-V_0)}}{\hbar} \longrightarrow \frac{\sqrt{2m|V_0|}}{\hbar}, \quad k \text{ fisso}.$$

$$k'a = \frac{\sqrt{2m|V_0|}}{\hbar} a = \frac{\sqrt{2mf}}{\hbar}\sqrt{a} \to 0$$

$\therefore \sin k'a \simeq k'a$, ma

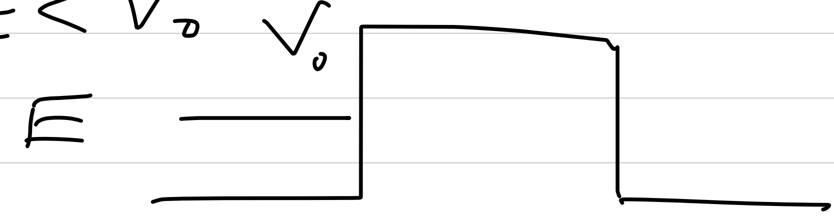
$$\left(k'^2 a = \frac{2mf}{\hbar^2} \right)$$

$$D \simeq \frac{4k^2}{4k^2 + k'^2(k'a)^2} = \frac{4k^2}{4k^2 + \frac{4m^2f^2}{(\hbar^2)^2}}$$

$$= \frac{1}{1+\alpha^2}, \quad \alpha \equiv \frac{mf}{k\hbar^2}$$

OK!

Effetto tunnel. $E<V_0$



$$\psi^{int} = Ce^{Kx} + De^{-Kx} \quad \text{ecc.} \Rightarrow$$

In realtà non è necessario ripetere il calcolo.

I passaggi che portano a

$$\tilde{F} = \frac{1}{\cos k'a - \frac{i}{2}\left(\frac{k'}{k}+\frac{k}{k'}\right)\sin k'a} \qquad (*)$$

$$B = \frac{\frac{i}{2}\left(\frac{k'}{k}-\frac{k}{k'}\right)\sin k'a}{\cos k'a - \frac{i}{2}\left(\frac{k'}{k}+\frac{k}{k'}\right)\sin k'a} \qquad (**)$$

Sono manipolazioni algebriche: non è stato usato il fatto che

$k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}$ sia reale.

Infatti, per $E<V_0$, $k'=iK$, $K=\frac{\sqrt{2m(V_0-E)}}{\hbar}$,

e i risultati (*) e (**) sono validi.

Usando

$$\sin k'a = \frac{1}{2i}\left(e^{ik'a} - e^{-ik'a}\right) = \frac{i}{2}\left(e^{\kappa a} - e^{-\kappa a}\right) \to i\,\mathrm{sh}\,\kappa a$$

$$\frac{k'}{k} = i\frac{\kappa}{k}, \quad \cos k'a = \frac{1}{2}\left(e^{\kappa a} + e^{-\kappa a}\right) = \mathrm{ch}\,\kappa a.$$

$$\therefore\ \tilde{E} = \frac{1}{\cos k'a - \frac{i}{2}\left(\frac{k'}{k} + \frac{k}{k'}\right)\sin k'a}$$

$$= \frac{1}{\mathrm{ch}\,\kappa a + \frac{i}{2}\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right)\mathrm{sh}\,\kappa a}$$

$$B = \frac{\frac{i}{2}\left(\frac{k'}{k} - \frac{k}{k'}\right)\sin k'a}{\cos k'a - \frac{i}{2}\left(\frac{k'}{k} + \frac{k}{k'}\right)\sin k'a} =$$

$$= \frac{-\frac{i}{2}\left(\frac{\kappa}{k} + \frac{k}{\kappa}\right)\mathrm{sh}\,\kappa a}{\mathrm{ch}\,\kappa a + \frac{i}{2}\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right)\mathrm{sh}\,\kappa a}$$

$$D = |E|^2 = \frac{4k^2\kappa^2}{4k^2\kappa^2\,\mathrm{ch}^2\kappa a + (\kappa^2 - k^2)^2\,\mathrm{sh}^2\kappa a}$$

$$= \frac{4k^2\kappa^2}{4k^2\kappa^2 + (\kappa^2 + k^2)^2\,\mathrm{sh}^2\kappa a}.$$

$$R = |B|^2 = \frac{(\kappa^2+k^2)^2 \operatorname{sh}^2 \kappa a}{4k^2\kappa^2 + (\kappa^2+k^2)^2 \operatorname{sh}^2 \kappa a}$$

(i) $D + R = 1$ ✓

(ii) $D \neq 0$ $(E < V_0)$: Effetto tunnel.

(iii) No effetto Ramsauer-Townsend.

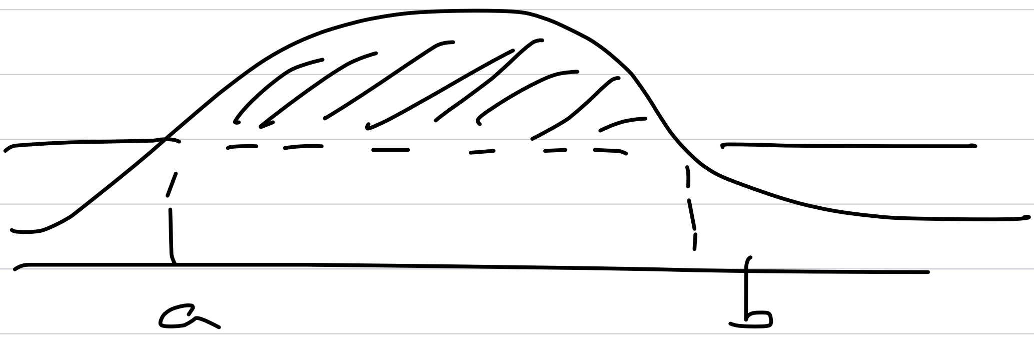
(iv) $D \to 0$, $V_0 \to \infty$

(v) V_0, a grandi. $\Rightarrow$

$$D \sim e^{-2\kappa a} \sim e^{-\frac{2}{\hbar}\int_a^b dx\,|p|}$$

→ "Azione classica"

$|p| \to \sqrt{2m(V_0 - E)}$



$D \to 0$, $\hbar \to 0$ OK.