Electrostatic induction in Coulomb experiment

In Coulomb's original experiment two charged metallic spheres were used instead of two point charges. If the surface charge densities of the spheres were uniform, the spheres would behave like point charges, and the force between them be given by the usual Coulomb's law

$$\mathbf{F} = k_0 \frac{q_1 q_2}{r^2} \hat{\mathbf{r}}, \tag{1}$$

where q_1 and q_2 are the charges on the spheres and $\mathbf{r} = r\hat{\mathbf{r}}$ is the distance between the centers. However, electric induction modifies the surface densities, so that a correction to (1) is needed. We expect the induction effects to be important if the radius a of the spheres is not negligibly small with respect to r.

a) Using the method of image charges, describe the solution for the electrical potential as a series expansion and identify the expansion parameter. For simplicity, assume the spheres to be identical and to have the same charge Q, as in Figure 1.

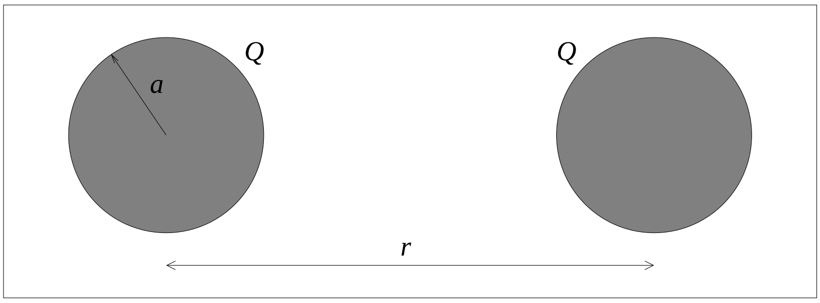

Figure 1:

b) Evaluate the lowest order correction to the force between the spheres with respect to Coulomb's law.

Solution

a)

As a zero order approximation, the problem is equivalent to having a point charge q in the centre of each sphere. Neglecting electrical induction effects, we would have $q = Q$ and the force would be the usual Coulomb's law between the two point charges.

At first order, each zero-order point charge q induces an image charge $q' = -(a/r)q$ located inside the other sphere, at a distance $a' = a^2/r$ from its center. On each sphere, the potential generated on its surface by the external charge q is canceled by the potential generated by the internal image charge. For symmetry reasons, we place two image charges q' at a distance a' from both spheres, as in Figure 2. Note that, at this stage, on each sphere we must have $q + q' = Q$.

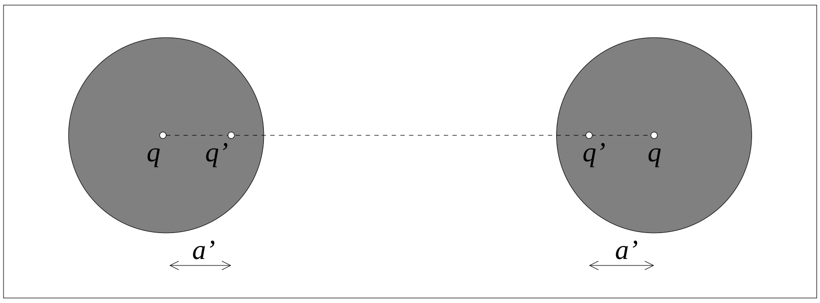

Figure 2:

At second order, we take into account, for instance, the effect of the image charge q' inside the left-side sphere on the right-side sphere. In order to keep the potential constant on the surface of the right-side sphere, we must introduce a second image charge q'' inside it. Since the distance of the left-side image q' from the center of the right-side sphere is $r - a'$, we must have $q'' = -q'a/(r - a') = q(a^2/r^2)/(1 - a^2/r^2)$. The new image charge q'' must be located at a distance $a'' = a^2/(r - a') = (a^2/r)/(1 - a^2/r^2)$ from the center of the right-side sphere, as in Figure 3. For symmetry reasons, a second order image charge must be introduced also in the left-side sphere.

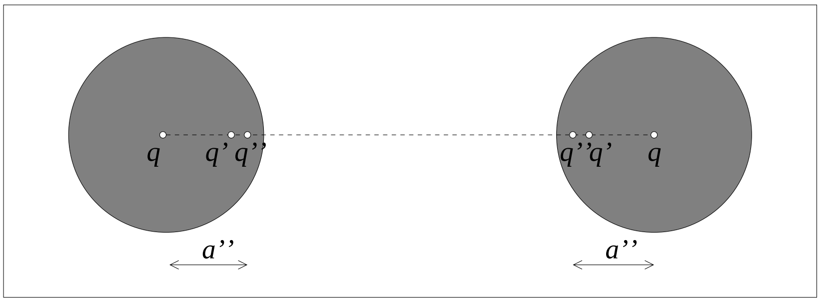

Figure 3:

Higher order approximations are obtained by iterating the procedure, obtaining a series of image charges $q, q', q'', q''', \ldots$ inside each sphere, each additional charge being of order $\sim (a/r)$ of the preceding one. Hence, the smaller the value of a/r, the sooner one may truncate the series obtaining a good approximation. Since the spheres are electrically isolated and the total charge is Q, we have the constraint $q + q' + q'' + \ldots = Q$.

b)

To determine the lowest order correction to Coulomb's law we consider only the first two image charges in the series (q and q'). Since $q' = -(a/r)q$ and $q + q' = Q$ to this order of approximation, we obtain

$$q = \frac{Q}{1 - a/r}, \qquad q' = -\frac{(a/r)Q}{1 - a/r}. \tag{2}$$

The force between the spheres can be computed as the force between the image charges, and it is thus composed of four terms. The first term is the force between the two charges q whose distance is R. The second and third term are the two identical contributions of the force between q and q', whose distance is $r - a'$. The fourth term is the force between the two charges q', whose distance is $r - 2a'$. Summing up all those contributions we obtain

$$\begin{aligned} F &= k_0 \left(\frac{q^2}{r^2} - 2\frac{qq'}{(r-a')^2} + \frac{q'^2}{(r-2a')^2} \right) \\ &= k_0 \frac{Q}{r^2} \frac{1}{(1-a/r)^2} \left(1 - \frac{2(a/r)}{(1-a^2/r^2)^2} + \frac{(a/r)^2}{(1-2a^2/r^2)^2} \right). \end{aligned} \tag{3}$$

We now expand F in a series of $x = a/r < 1$, up to third order. Using the expansions

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots, \qquad (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \tag{4}$$

so that

$$(1-x^2)^{-2} = 1 + 2x^2 + \mathcal{O}(x^4), \qquad (1-2x^2)^{-2} = 1 + 4x^2 + \mathcal{O}(x^4) \tag{5}$$

we obtain

$$\begin{aligned} \frac{1}{(1-x)^2} \left(1 - \frac{2x}{(1-x^2)^2} + \frac{x^2}{(1-2x^2)^2} \right) &= \left(1 + 2x + 3x^2 + 4x^3\right)\left(1 - 2x - 4x^3 + x^2\right) + \mathcal{O}(x^4) \\ &= 1 - 4x^3 + \mathcal{O}(x^4) \end{aligned} \tag{6}$$

since the terms of order x and x^2 vanish. The force between the spheres can thus be written as

$$F = k_0 \frac{Q}{r^2} \left(1 - 4\frac{a^3}{r^3} \right) \tag{7}$$

This result can be understood in terms of the multipole expansion of the field. The first two multipole terms of the charge distribution on each sphere are a monopole equal to the total charge Q and an electric dipole $\mathbf{p} = q'a'\hat{\mathbf{r}}$ (with respect to the center of the sphere) where $\hat{\mathbf{r}}$ is oriented towards the center of the opposite sphere. On this latter, the dipole moment generates an electric field

$$\mathbf{E}_p = k_0 \frac{3(\hat{\mathbf{r}} \cdot \mathbf{p})\hat{\mathbf{r}} - \mathbf{p}}{r^3}, \qquad E_p = \mathbf{E}_p \cdot \hat{\mathbf{r}} = -2k_0 \frac{p}{r^3} \simeq -2k_0 Q \frac{a^3}{r^5}, \tag{8}$$

higher order terms having been neglected in the last equality. The dipole field thus leads to a force term between the spheres $F_p = 2QE_p \simeq -4k_0 Q(a^3/r^5)$, which is just the second term in Eq.(7), the first term obviously corresponding to the monopole interaction.

¿From Eq.(7) we find that a ratio $a/r \simeq 0.13$ is enough to reduce the systematic deviation from Colulomb's law below 1%.