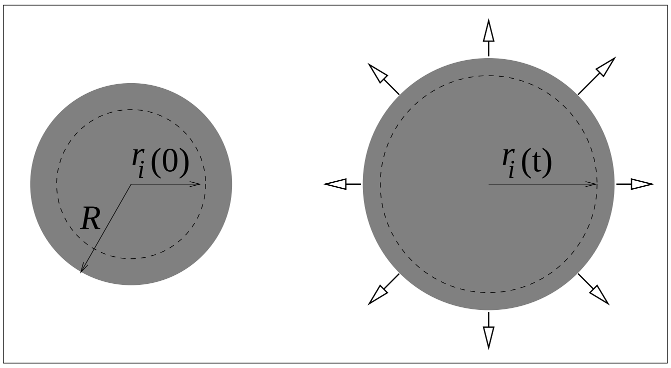
Coulomb explosions

A spherical cloud of radius R and total charge Q contains N particles of individual charge $q = Q/N$ and mass m. At the initial time ($t = 0$), the particle density is uniform.
a) Find the potential energy of a test charge located a distance $r < R$ from the cloud center.



Due to the Coulomb repulsion, the cloud begins to expand in radial direction, keeping its spherical symmetry. The particles do not overturn each other during their motion, i.e. if two particle layers are initially located at the positions $r_1(0)$ and $r_2(0) > r_1(0)$ respectively, then $r_2(t) > r_1(t)$ holds at any $t > 0$.
b) Discuss the validity of the above assumptions.
c) Let $r_i = r_i(t)$ be the position at time t of those particles which at $t = 0$ are located at the $r_{i0} = r_i(0) < R$ position (i.e. $r_i(t)$ defines an infinitesimally thin, expanding shell): see figure . Show that the equation of motion for $r_i(t)$ is

$$m\frac{d^2 r_i}{dt^2} = \frac{qQ}{4\pi\epsilon_0 r_i^2}\left(\frac{r_{i0}}{R}\right)^3 \tag{1}$$

d) Find the initial position of those particles which acquire the maximum kinetic energy during the cloud expansion, and determinate the value of such maximum energy.
e) Find the energy spectrum, i.e. the distribution of the particles as a function of their final kinetic energy. Compare the total kinetic energy with the potential energy initially stored in the electrostatic field.
f) Show that the cloud density remains uniform during the expansion.

Solution

a) The electric field has radial symmetry, $\mathbf{E} = E(r)\hat{\mathbf{r}}$. The value of $E(r)$ can be found from Gauss theorem, being $4\pi r^2 E(r) = Q_{int}(r)/\epsilon_0$, where $Q_{int}(r)$ is the charge inside the sphere of radius r. Thus $Q_{int} = Q(r/R)^3$ for $r < R$ and $Q_{int} = Q$ for $r > R$. We thus obtain

$$E(r) = \frac{Q}{4\pi\epsilon_0} \times \begin{cases} r/R^3 & (r < R) \\ 1/r^2 & (r > R) \end{cases} \tag{2}$$

Since $E = -\partial V/\partial r$, the potential is obtained via a simple integration:

$$V(r) = \frac{Q}{4\pi\epsilon_0} \times \begin{cases} (-r^2/2R^3 + 3/2R) & (r < R) \\ 1/r & (r > R) \end{cases} \tag{3}$$

The constant has been chosen to ensure the potential to be continuous and to vanish for $r \to \infty$. The potential energy of a *test* charge q_t located at distance r from the centre is $q_t V(r)$. Under the action of the electric field, the test charge would move and convert all its potential energy into kinetic energy *if the field remains stationary during the charge motion*, i.e. *if all the source charges of the field remain fixed*.

b) At $t = 0$ the electric field inside the spherical cloud increases with r. Thus, the "outer" particles located at larger r will have a higher acceleration that the "inner" particles located at smaller r, and after an infinitesimal time interval the "outer" particles will have a higher velocity (within the assumption that all particles have zero velocity at $t = 0$) and will not be overturned by the "inner" ones. Moreover, the acceleration has also radial symmetry and thus any spherical layer preserves its shape. These arguments can be iterated for any following time, thus the particles do not overturn each other and the spherical symmetry is preserved.

c) Since the particles do not overturn each other, the charge inside a sphere of radius $r_i(t)$ is a constant. Thus, applying Gauss theorem to the spherical surface of radius $r = r_i(t)$, i.e. the surface delimited by the layer of particles which at $t = 0$ were located at $r = r_{i0} = r_i(t = 0)$, yields

$$4\pi r_i^2 E(r_i) = \frac{Q}{\epsilon_0}\left(\frac{r_{i0}}{R}\right)^3. \tag{4}$$

Thus the field on the particles is

$$E(r_i) = \frac{Q}{4\pi\epsilon_0 r_i^2}\left(\frac{r_{i0}}{R}\right)^3 \tag{5}$$

from which Eq.(1) is obtained. The force on the particles, and thus their acceleration, increases with r_{i0}, which is consistent with the "no-overturning" assumption.

d)

Under the action of the force (1) each charge layer moves from the initial position $r_i(0) = r_{i0}$ to $r_i = \infty$. The final kinetic energy is obtained by calculating the work done by the force:

$$K = K(r_{i0}) = \int_{r_{i0}}^{+\infty} \frac{qQ}{4\pi\epsilon_0 r_i^2}\left(\frac{r_{i0}}{R}\right)^3 dr_i = \frac{qQ}{4\pi\epsilon_0 r_{i0}}\left(\frac{r_{i0}}{R}\right)^3 = \frac{qQ}{4\pi\epsilon_0}\left(\frac{r_{i0}^2}{R^3}\right). \tag{6}$$

$K(r_{i0})$ is a monotonically growing function of r_{i0} and thus its maximum value is obtained for $r_{i0} = R$:

$$K_{\max} = K(R) = \frac{qQ}{4\pi\epsilon_0 R}. \tag{7}$$

This means that the particles initially located at $r_{i0} = R$, i.e. at the cloud surface, are those which acquire the maximum kinetic energy.

e) By definition the energy distribution or spectrum is the number of particles per infinitesimal energy interval:

$$f(K) = \frac{dN}{dK} = \frac{dN/dr_{i0}}{dK/dr_{i0}}. \tag{8}$$

Here dN is the number of particles between $K = K(r_{i0})$ and $K + dK$, that is equal to the number of particles which at $t = 0$ are in the shell between r_{i0} and $r_{i0} + dr_{i0}$:

$$dN = n(0)4\pi r_{i0}^2 dr_{i0} = 3N\frac{r_{i0}^2}{R^3}dr_{i0}. \tag{9}$$

Thus we obtain

$$f(K) = \frac{3Nr_{i0}^2/R^3}{2r_{i0}qQ/4\pi\epsilon_0 R^3} = \frac{3Nr_{i0}}{2qQ/4\pi\epsilon_0} = \frac{3N}{2}\frac{K^{1/2}}{K_{\text{max}}^{3/2}}, \qquad K \le K_{\text{max}}. \tag{10}$$

The total energy is given by

$$K_{\text{tot}} = \int_0^{K_{\text{max}}} Kf(K)dK = \frac{3N}{2}\int_0^{K_{\text{max}}} \left(\frac{K}{K_{\text{max}}}\right)^{3/2} = \frac{3N}{5}K_{\text{max}} = \frac{3N}{5}\frac{qQ}{4\pi\epsilon_0 R} \tag{11}$$

which is the total electrostatic energy of the charged sphere: all the electrostatic energy stored in the initial configuration is eventually converted into kinetic energy.

As a frequent mistake, one could suppose that the final kinetic energy of the particles in the shell of initial radius r_{i0} would be equal to the potential energy of such particles at $t = 0$, i.e. $qV(r_{i0})$ where V is given in Eq.(3). This is evidently wrong because actually the particles with the highest value of $V(r_{i0})$ get the lowest kinetic energy (zero)! The point is that the field is electrostatic ($\boldsymbol{\nabla} \times \mathbf{E} = 0$) at any time, but it is time dependent; thus, V can be defined for any value of t, but it can *not* be used to estimate the final kinetic energy because V changes as the particles move.

Only for the particles starting from $r_{i0} = R$, i.e. the most energetic ones, the gain of kinetic energy equal the initial potential energy $qV(R, t = 0)$: this is because, incidentally, those particles (and only those) are accelerated by a field that can be considered as static, as it is simply the field of a point charge Q located at $r = 0$.

f) Introducing the new variable $x(t) = r_i(t)/r_i(0)$, Eq.(1) can be rewritten as

$$m\frac{d^2x}{dt^2} = \frac{qQ}{4\pi\epsilon_0 R^3 x^2}, \tag{12}$$

where r_{i0} does not appear anymore; this means that the solution of Eq.(12) $x = x(t)$ with the initial condition $x(0) = 1$ describes the motion of all the charge layers. As a consequence, for two layers (labeled 1 and 2) initially located at $r_1(0)$ and $r_2(0) > r_1(0)$, it follows that $r_2(t) > r_1(t)$, i.e. the layer do not overturn. The number of particles contained between the layers 1 and 2 is a constant and it is given by

$$\delta N_{12} = N\left[\left(\frac{r_2(0)}{R}\right)^3 - \left(\frac{r_1(0)}{R}\right)^3\right] \tag{13}$$

and thus the density between the two layers at the time t is given by

$$n(t) = \frac{\delta N_{12}}{(4\pi/3)[r_2^3(t) - r_1^3(t)]} = \frac{(N/R^3)[r_2^3(0) - r_1^3(0)]}{(4\pi/3)[r_2^3(0) - r_1^3(0)]x^3(t)} = \frac{n(0)}{x^3(t)}\,, \tag{14}$$

being $n(0) = N/(4\pi R^3/3)$ the initial density. This result does not depend on the particular choice of the two layers, therefore the density is uniform at any t and decreases with time as $x^{-3}(t)$.