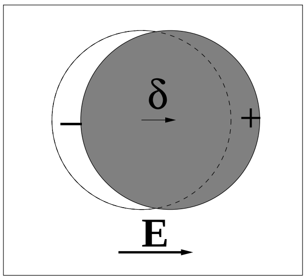
A metallic sphere in an external field. Mie oscillations



A metallic sphere of radius R is made up of ions and electrons with charge $+Ze$ and $-e$, respectively, and uniform particle density n_i and n_e, respectivity. The total charge is zero.
The sphere is placed in a constant, uniform electric field $\mathbf{E}$. We want to determine the charge distirbution on the sphere and the related electric field within the assumption that due to the external field all the electrons are displaced by $-\boldsymbol{\delta}$ with respect to ions, with $\delta \ll R$, and that the ion and electron spheres keep their shape.
a) Find the electrostatic field in the region where the ion and electron spheres overlap.
b) Find the electrostatic field in the region outside both the ion and electron spheres.
c) Give a numerical estimate of δ for a metallic sphere in an external field of intensity $E = 10^3$ V/m.
d) Assuming, following response **c)**, that δ is small enough to assume the charge density as distributed on the surface of the sphere, find the surface charge density.
The external field is now removed within a negligible time.
e) Find the electrostatic energy of the polarized sphere as a function of δ.
f) Without the external field, the electrons start to oscillate around their equilibrium positions. Find the oscillation frequency.

Solution

a) Inside a conductor the total electric field must be zero. Thus the charge redistribution induced by the external field must generate a field $\mathbf{E}_{int} = -\mathbf{E}$ inside the sphere, such that the total field is $\mathbf{E}_{int} + \mathbf{E} = 0$. We must thus verify that the rigid displacement of the electrons produces an uniform field inside the sphere.

Within our starting assumptions, due to the superposition principle the electrostatic field at any point can be computed as the sum of the fields generated by the ion sphere (having charge density $\rho_i = Zen_i \equiv \rho$) and by the electron sphere (having charge density $\rho_e = -en_e = -\rho$).

The field generated inside an uniformly charged sphere with charge density ρ is radial and is given by $\mathbf{E}(\mathbf{r}) = \rho\mathbf{r}/3\epsilon_0$. Thus, the two spheres generate the fields $\mathbf{E}_\pm = \pm\rho\mathbf{r}_\pm/3\epsilon_0$ respectively, where $\mathbf{r}_\pm$ are the position vectors of the centres of the two spheres, which we assume to be located on the x–axis at the points $+\boldsymbol{\delta}/2$ e $-\boldsymbol{\delta}/2$. We thus write $\mathbf{r}_\pm = \mathbf{r} \pm \boldsymbol{\delta}/2$. The total field in the overlap region can be written as

$$\mathbf{E}_{int} = +\frac{\rho}{3\epsilon_0}\left(\mathbf{r} - \frac{\boldsymbol{\delta}}{2}\right) - \frac{\rho}{3\epsilon_0}\left(\mathbf{r} + \frac{\boldsymbol{\delta}}{2}\right) = -\frac{\rho}{3\epsilon_0}\boldsymbol{\delta}\,. \tag{1}$$

The field $\mathbf{E}_{int}$ is thus *uniform* and proportional to $-\boldsymbol{\delta}$. Posing $\mathbf{E}_{int} = -\mathbf{E}$ we obtain $\delta = 3\epsilon_0 E/\rho$.

b) The electrostatic field generated *outside* a charged sphere is the field of a point charge $Q = \rho V$ where $V = 4\pi R^3/3$. Thus, the electrostatic field in the outer region (outside both spheres) is the sum of the field of two point charges $\pm Q$, which gives, if $R \gg \delta$, the field of a dipole $\mathbf{p} = Q\boldsymbol{\delta}$. In a system of polar coordinates (r, θ, ϕ) with respect to the axis parallel to $\mathbf{p}$, the dipole field can be written as

$$\mathbf{E}_{dip} = \mathbf{E}(r, \theta) = k_0\frac{3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}}{r^3}\,,$$

where $\mathbf{r} = r\hat{\mathbf{n}}$ and $k_0 = 1/4\pi\epsilon_0$. The total field outside the spheres is $\mathbf{E} + \mathbf{E}_{dip}$. Inserting the value of $\boldsymbol{\delta}$ from the answer **a)** we can write

$$\mathbf{E}_{tot} = \mathbf{E} + [3\hat{\mathbf{n}}(\mathbf{E}\cdot\hat{\mathbf{n}}) - \mathbf{E}]\left(\frac{R}{r}\right)^3 \qquad (r > R).$$

c) Due to Gauss theorem, at a surface of discontinuity between two regions (1 and 2) the perpendicular components of the electric field are related at any point on the surface by

$$E_{2\perp} - E_{1\perp} = \frac{\sigma}{\epsilon_0}\,,$$

wehre σ is the surface charge density. In our case we label the inner region as 1 and the outer region as 2. Thus, in the region 1 the field is zero, while in the region 2 the field is the sum of the external field and the induced dipole field. At a point located at the radius $r = R$ and at an angle θ with respect to $\mathbf{E}$, we have

$$E_{2\perp} = \mathbf{E}\cdot\hat{\mathbf{n}} + \mathbf{E}_{dip}(r = R)\cdot\hat{\mathbf{n}} = \mathbf{E}\cdot\hat{\mathbf{n}} + 3\mathbf{E}\cdot\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) - \mathbf{E}\cdot\hat{\mathbf{n}} = 3\mathbf{E}\cdot\hat{\mathbf{n}} = 3E\cos\theta\,,$$

and thus we obtain

$$\sigma = \sigma(\theta) = \epsilon_0(E_{2\perp} - E_{1\perp}) = 3\epsilon_0 E\cos\theta = \rho\delta\cos\theta\,. \tag{2}$$

d) In a metal $n_e = Z(\rho_m/M)N_A$ where Z is the valence, ρ_m is the mass density, M is the atomic weight, and $N_A = 6 \times 10^{23}$ is the Avogadro number. Typically $n_e \sim 10^{28}$ m^{-3} and $\rho = en_e \sim 10^9$ C m^{-3} being $e = 1.6 \times 10^{-19}$ C. We thus estimate $\delta = 3\epsilon_0 E/\rho \sim 10^{-17}$ m if $E = 10^3$ V/m (a rather strong field). The value obtained for δ is smaller by orders of magnitude than the spacing between atoms in a crystalline lattice ($\sim 10^{-10}$ m); it makes then sense to consider the charge as being distributed on the surface. Formally, this is equivalent to tale the limits $\delta \to 0$ and $\rho \to \infty$ in order for $\sigma = \rho\delta$ to keep a finite value.

e) To find the electrostatic energy it is convenient to use the formula

$$U_{es} = \int dV \frac{\epsilon_0}{2} E^2 .$$

In the inner region the field is uniform and we immediately find

$$U_{es}^{(int)} = \frac{\epsilon_0}{2}\left(\frac{\rho\delta}{3\epsilon_0}\right)^2 V.$$

The contribution from the outer region, being

$$E^2 = (k_0 p/r^3)^2 (3\cos^2\theta + 1)\,, \qquad dV = r^2 \sin\theta dr d\theta d\phi\,,$$

is given by

$$U_{es}^{(ext)} = 2\pi \frac{\epsilon_0}{2} k_0^2 p^2 \int_R^{+\infty} r^2 dr \int_0^{\pi} \sin\theta d\theta \frac{3\cos^2\theta + 1}{r^6}$$

and with some simple calculations we obtain

$$U_{es}^{(ext)} = \frac{p^2}{12\pi\epsilon_0 R^3} = \frac{(\rho\delta)^2}{9\epsilon_0}\left(\frac{4\pi}{3}R^3\right).$$

Rearranging these expressions we obtain $U_{es}^{(int)} = U_{es}^{(ext)}/2$ and

$$U_{tot} = U_{es}^{(int)} + U_{es}^{(ext)} = \frac{(\rho\delta)^2}{6\epsilon_0} V = \frac{V(en_e\delta)^2}{6\epsilon_0}.$$

f) The force on the electron sphere is minus the derivative of the electrostatic energy with respect to the displacement:

$$\mathbf{F} = -\frac{\partial U_{tot}}{\partial \boldsymbol{\delta}} = -\frac{V(en_e)^2}{3\epsilon_0}\boldsymbol{\delta}.$$

The equation of motion is $M\ddot{\boldsymbol{\delta}} = \mathbf{F}$ where $M = m_e n_e V$. Thus

$$\ddot{\boldsymbol{\delta}} = -\frac{n_e e^2}{3\epsilon_0 m_e}\boldsymbol{\delta} \equiv -\frac{\omega_p^2}{3}\boldsymbol{\delta}\,,$$

where $\omega_p \equiv \sqrt{n_e e^2/\epsilon_0 m_e}$.

We may obtain the same equation using Eq.(1) and the fact that all the electrons oscillate in phase with the same displacement $\boldsymbol{\delta}$:

$$m_e\ddot{\boldsymbol{\delta}} = -e\mathbf{E}_{int} = -e\frac{en_e}{3\epsilon_0}\boldsymbol{\delta} = -m_e\frac{\omega_p^2}{3}\boldsymbol{\delta}\,.$$

Thus, the electrons perform an oscillatory motion:

$$\boldsymbol{\delta}(t) = \boldsymbol{\delta}(0)\cos\left(\frac{\omega_p t}{\sqrt{3}}\right)\,.$$

The frequency ω_p is called the *plasma frequency* and is a characteristic parameter of a metal, since it only depends on the free electron density n_e. The electron sphere oscillates with frequency $\omega_p/\sqrt{3}$. This phenomenon is known as "Mie oscillation" and $\omega_p/\sqrt{3}$ is also known as the "Mie frequency".