An artificial dielectric

Consider a insulating liquid (having negligible dielectric susceptibility) in which metallic nanospheres have been immersed. Each nanosphere has radius a and the nummber of nanospheres per unit volume is n.
a) Find the dielectric susceptibility χ of the system as a function of the fraction of the volume filled by the nanospheres, in the mean field approximation (MFA).
b) The MFA requires the field generated by a single nanosphere on the nearest neighbour to be much smaller than the mean field due to the collective contribution of all the nanosphere. Derive a condition on n and a for the validity of the MFA.

Solution

a) A metallic sphere in an uniform field acquires a dipole

$$\mathbf{p} = a^3 4\pi\varepsilon_0 \mathbf{E} = 3V\varepsilon_0 \mathbf{E}, \tag{1}$$

where $V = \frac{4}{3}\pi a^3$ is the volume of the sphere. Writing $\mathbf{P} = \varepsilon_0 \chi \mathbf{E} = n\mathbf{p}$, we obtain

$$\mathbf{P} = 3\varepsilon_0 nV\mathbf{E} \tag{2}$$

from which $\chi = 3f$, where $f = nV$ is the fraction of the unit volume filled by the nanospheres.
b) The average distance ℓ between two spheres is of the order of $1/\sqrt[3]{n}$. At a distance ℓ the electric field of a dipole is of the orfer of

$$E_d \approx \frac{1}{4\pi\varepsilon_0}\frac{p}{\ell^3} = \frac{1}{4\pi\varepsilon_0} a^3 4\pi\varepsilon_0 En = a^3 nE. \tag{3}$$

Thus, the condition $E_d \ll E$ requires $na^3 \ll 1$.