Atmospheric Electricity

In a very rough model of atmospheric electricity,[1] the charge distribution inside a cloud during a storm is approximated by two point charges $+Q$ and $-Q$, located on the same vertical axis at heights h_p and h_n, respectively. Typical values are $Q = 40$ C, $h_n = 5$ km, and $h_p = 10$ km.

Consider the earth as a conductor and neglect its curvature.

a) Calculate the numerical value of the electric field at the Earth's surface, on the vertical axis along which the charges are located.

b) Calculate the electrostatic energy of the charge distribution.

c) Find the expression of the electric field generated at the Earth surface, showing that there is a point where the field is zero, and give the distance between the point and the vertical axis.

[1] See e.g. M. Uman, *The lightning discharge* (Dover, 2001).

Solution

a) In the proposed model, the Earth is an infinite conductor. The electrostatic potential must vanish at the Earth surface ($y = 0$). We can find the solution for the potential using the method of image charges, placing two imaginary charges $-Q$ and $+Q$ on the same axis (y) of the real charges, at the positions $y = -h_n$ and $y = -h_p$ (i.e. under the Earth's surface), respectively.

Summing up all the contributions from the four charges to the electric field in the point ($\rho = 0, y = 0$) we obtain

$$\begin{aligned} E(\rho = 0, y = 0) &= k_0 Q \left(-\frac{1}{h_p^2} + \frac{1}{h_n^2} + \frac{1}{h_n^2} - \frac{1}{h_p^2} \right) = 2k_0 Q \left(\frac{1}{h_n^2} - \frac{1}{h_p^2} \right) \\ &\simeq 2.2 \times 10^4 \text{ V/m} \end{aligned}$$

($k_0 = 1/4\pi\epsilon_0 \simeq 9 \times 10^9$).

b) To find the electrostatic energy we start from the expression from the potential energy of a pair of charges located in the $\mathbf{r}_1$, $\mathbf{r}_2$ positions:

$$U(\mathbf{r}_1 - \mathbf{r}_2) = k_0 \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} . \tag{1}$$

If all the charges of the configuration were real, the potential energy would be given by Eq.(1) summed over all the couples of charges. In our case the potential energy is half of this value. In fact, if we move all the charges to infinite distance, the total work done is the sum of all the individual work $\mathcal{L}_i$ done by the forces on each charge q_i. If the image charges q_i' were replaced by real charges, for symmetry reasons the work done on q_i' would be equal to the work done on q_i (because both the force and the displacement change sign when comparing q_i with q_i') and the sum of individual works would double, giving Eq.(1) as the result. Thus, the potential energy is half of Eq.(1):

$$\begin{aligned} U_{tot} &= \frac{1}{2} \left(-2 \times k_0 \frac{Q^2}{h_p - h_n} - k_0 \frac{Q^2}{2h_p} - k_0 \frac{Q^2}{2h_n} + 2 \times k_0 \frac{Q^2}{h_p + h_n} \right) \\ &= -k_0 Q^2 \left(\frac{1}{h_p - h_n} + \frac{1}{4h_p} + \frac{1}{4h_n} + \frac{1}{h_p + h_n} \right) . \end{aligned}$$

In other words, when summing over all the couples of charges, for a "real–image" couple we must divide (1) by 2, while we must not take into account any "image–image" couple.

A numerical estimate gives

$$U_{tot} \simeq 3 \times 10^9 \text{ J} .$$

c) The computation is made easier by observing that at the surface the electric field is in the perpendicular direction (i.e. along y), thus we just need to consider the y component of the field generated by each charge. Let $\mathbf{r}_i$ be the distance from the i-th charge (located at $y = y_i$) from the circumference of radius ρ around the y-axis. We have

$$E_{i,y} = \left(k_0 \frac{q}{r_i^3} \mathbf{r}_i \right) \cdot \hat{\mathbf{y}} = k_0 \frac{q}{(y_i^2 + \rho^2)^{3/2}} y_i .$$

We now sum all the contributions, taking into account that each real charge gives the same contribution as its image:

$$E_y(\rho) = -2k_0Q\frac{h_p}{(h_p^2+\rho^2)^{3/2}} + 2k_0Q\frac{h_n}{(h_n^2+\rho^2)^{3/2}}\,.$$

Posing $E_y(\rho) = 0$ we obtain the equation

$$\frac{\rho^2+h_p^2}{h_p^{2/3}} = \frac{\rho^2+h_n^2}{h_n^{2/3}}\,,$$

from which we get

$$\rho = (h_p h_n)^{1/3}(h_p^{2/3}+h_n^{2/3})^{1/2} \simeq 28\ \text{km}\,.$$