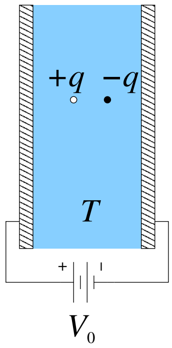
Debye screening



A ionized gas (plasma), composed of two charged species having opposite charges $+q$ and $-q$, is contained between two parallel metallic plates. The gas is globally neutral, i.e. the number of positive and negative charges are equal. A generator keeps the potential difference between the plates equal to V_0.
The system is in thermal equilibrium at the temperature T; we thus assume that both the particle densities of the two species, n_+ and n_-, are a function of the potential energy according to Boltzmann's distribution:

$$n_{\pm} = n_0 \exp\left(\frac{-U_{\pm}}{k_B T}\right), \tag{1}$$

where U_+ and U_- are the potential energies of the two species and k_B is Boltzmann's constant.
Find the electrostatic potential and the field between the plates, assuming that all the potential energy is of electrostatic nature and it is small compared to the thermal energy ($|U_{\pm}| \ll k_B T$), and neglecting boundary effects.

Solution

Let $V = V(x)$ be the electrostatic potential between the plates, which are located at $x = \pm a/2$. The potential energy of the two species is given by $U_\pm = \pm qV$. Using Eq.(1) we may thus write the charge density as a function of V:

$$\rho = \rho(x) = qn_+ - qn_- = qn_0 \left(e^{-qV/k_BT} - e^{qV/k_BT}\right) .$$

Using Poisson's equation $\nabla^2 V = \rho/\epsilon_0$, we obtain a nonlinear equation for V:

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0} = -\frac{qn_0}{\epsilon_0}\left(e^{-qV/k_BT} - e^{+qV/k_BT}\right) .$$

This is a second rank equation, thus two boundary conditions are needed to determine a solution. As a first condition the potential drop between the plates is fixed:

$$V(a/2) - V(-a/2) = V_0 . \tag{2}$$

As a second condition, the system is globally neutral:

$$0 = \int_{-a/2}^{+a/2} \rho dx = -\epsilon_0 \int_{-a/2}^{+a/2} \partial_x^2 V dx = -\epsilon_0 \partial_x V|_{-a/2}^{+a/2} ,$$

from which we obtain a condition of the first derivatives of V

$$\frac{dV}{dx}(+a/2) = \frac{dV}{dx}(-a/2) . \tag{3}$$

Within the approximation $qV \ll k_BT$, we can expand the exponential up to the first order:

$$\exp\left(\pm\frac{qV}{k_BT}\right) \simeq 1 \pm \frac{qV}{k_BT} .$$

The equation for V thus becomes

$$\frac{d^2V}{dx^2} \simeq -\frac{2q^2n_0}{\epsilon_0 k_BT}V \equiv \frac{1}{\lambda_D^2}V ,$$

where we introduced the length

$$\lambda_D \equiv \sqrt{\frac{\epsilon_0 k_BT}{2q^2n_0}} .$$

The generic solution is

$$V(x) = V_1 e^{x/\lambda_D} + V_2 e^{-x/\lambda_D} ,$$

where the two arbitrary constants V_1 and V_2 are to be determined from the boundary conditions. From Eq.(3) we obtain

$$V_1 e^{a/2\lambda_D} - V_2 e^{-a/2\lambda_D} = V_1 e^{-a/2\lambda_D} - V_2 e^{a/2\lambda_D} ,$$

which yields $V_1 = -V_2$. Thus

$$V(x) = V_1(e^{x/\lambda_D} - e^{-x/\lambda_D}) = 2V_1 \sinh\left(\frac{x}{\lambda_D}\right) ,$$

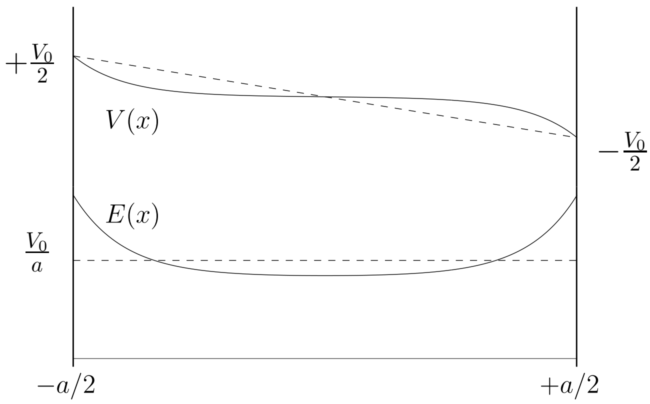
and using Eq.(2) in the form $V(\mp a/2) = \pm V_0/2$ we finally obtain

$$V(x) = -\frac{V_0}{2\sinh(a/2\lambda_D)} \sinh\left(\frac{x}{\lambda_D}\right) .$$

The electric field is given by

$$E = -\frac{dV}{dx} = \frac{V_0}{2\lambda_D} \frac{\cosh(x/\lambda_D)}{\sinh(a/2\lambda_D)} .$$

The figure shows the spatial profiles of $V(x)$ and $E(x)$ (for the case $a/\lambda_D = 10$) compared with the profiles we would obtain in the absence of the ionized medium. The field in the central region is lower than the value in vacuum. This is due to the screening effect from the charges accumulated near the surface of the plates.



Screening effects of a similar type occur for electrolites (the theory has been introduced by Debye and Hückel) and for plasmas in thermal equilibrium. We notice that the mass of the charged species does not appear, thus the model may be applied both to an electrolite with positive and negative ions and to a plasma with ions and electrons.