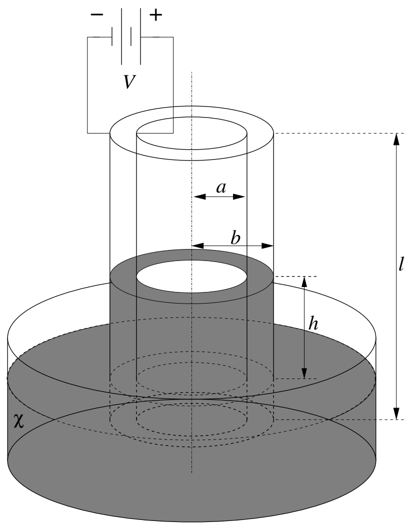
Measuring the dielectric constant of a liquid

A cylindrical capacitor has internal radius a, external radius $b > a$, and length $\ell \gg b$ such that boundary effects are negligible. The capacitor is partly immersed (along the vertical direction) in a liquid having mass density ρ and dielectrical suscettivity χ, in the presence of the gravitational field. When a generator maintains a potential difference V between the plates, the liquid is found to rise up in the capacitor for a length h, in steady conditions.



Show how from the measurement of h the value of χ can be obtained. (This is problem 4.13 of Jackson, *Classical Electrodynamics*, 3rd Edition).

Solution

Let us evaluate the variation of the energy of the system when the liquid rises by an amount h. The variation of electrostatic energy is obtained simply by observing that the capacitor partly filled with the liquid can be considered as two cylindrical capacitors in parallel, having the same radii and lengths h and $\ell - h$, respectively. Due to the rise of the liquid the capacity of the "lower" capacitor changes from the value $C_h = 2\pi h\epsilon_0/\ln(b/a)$ to the value $C'_h = \epsilon_r C_h$, where $\epsilon_r = 1 + \chi$, while the capacity of the "upper" capacitor does not change. Since the voltage V is a constant, the electrostatic energy changes by an amount

$$\Delta U_{es} = \frac{1}{2}\Delta C V^2 = \frac{1}{2}(C'_h - C_h)V^2 = \frac{\pi\epsilon_0 h\chi}{\ln(b/a)}V^2\,.$$

Due to the change of capacity at constant voltage, the charge on the plates change, and work is done by the generator to carry charges from one plate to the other. To displace an infinitesimal charge amount dQ the work done is $dW = V dQ$, thus the internal energy of the generator changes by the amount $dU_{gen} = -dW = -V dQ = -V d(CV) = -V^2 dC$. Since $dU_{es} = d(CV^2/2) = V^2 dC/2$, we find $dU_{gen} = -2dU_{es}$ and $dU_{es} + dU_{gen} = -dU_{es}$.

We now evaluate the change of gravitational energy due to the rise of the liquid. The potential energy of a circular corona of infinitesimal thickness dz is $dU_g = zg dm$, where $dm = \pi(b^2 - a^2)dz\rho$. Thus, the total variation in U_g is

$$\Delta U_g = \pi(b^2 - a^2)dz\rho g\int_0^h z\,dz = \frac{\pi}{2}(b^2 - a^2)dz\rho g h^2\,.$$

The *total* energy variation is thus given by

$$\Delta U_{tot} = \Delta U_{es} + \Delta U_{gen} + \Delta U_g = -\frac{\pi\epsilon_0 h\chi}{\ln(b/a)}V^2 + \frac{\pi}{2}(b^2 - a^2)\rho g h^2\,.$$

Differentiating with respect to h we obtain the equilibrium condition

$$\frac{\partial\Delta U_{tot}}{\partial h} = -\frac{\pi\epsilon_0\chi}{\ln(b/a)}V^2 + \pi(b^2 - a^2)\rho g h \doteq 0\,,$$

from which χ can be obtained as a function of h and of the other known parameters:

$$\chi = \frac{(b^2 - a^2)\rho g h\ln(b/a)}{\epsilon_0 V^2}\,.$$