Eddy currents in a solenoid

A long solenoid is made by a coil of n loops per unit length wrapped around a Ferrite cylinder of radius R and length $\ell \gg a$. Ferrite has magnetic permittivity μ_r and electrical conductivity σ. An AC current $I = I_0 \cos \omega t$ flows in the coils.
a) Find the *electric* field inside the solenoid.
b) Explain why the Ferrite cylinder warms up and find the dissipated power inside the cylinder.
c) Find how the induced currents modify the magnetic field produced by the coils.
(Boundary effects and the displacement current are assumed to be negligible).

Solution

a) If the current in the loops was time-independent (DC), neglecting boundary effects the **H** field would be uniform and parallel to the axis of the solenoid and given by $H = nI$ for $r < a$ (while $H = 0$ in the external region $r > a$). The field $\mathbf{B} = \mu\mathbf{H}$.

Since the current is time-dependent, the magnetic field is also time-dependent and thus induces a time-dependent electric field, which in turn induces a magnetic field, and so on ... In principle one has to find a solution to the complete set of Maxwell's equations, but in many cases of practical interest the slowly varying current approximation (SVCA) can be assumed. In the SVCA, we start by calculating **H** and **B** as in the magnetostatic case, and we use such expressions as source terms for the electric field (and related current density); then we use the electric field and current density obtained in this way as a source for corrections to the "static" components of the magnetic fields, checking *a posteriori* when such terms are a small corrections to the leading ones.

Following the SVCA procedure, we start with the magnetic field $\mathbf{B}_0 = \hat{\mathbf{z}}\mu n I_0 \cos\omega t$ and find the electric field $\mathbf{E}_1$ induced by $\mathbf{B}_0$ because of Faraday's law. Due to the cylindrical symmetry of the system, the field lines of the induced electric field are circular and centered on the axis, i.e. $\mathbf{E}_1 = E_1(r,t)\hat{\boldsymbol{\phi}}$. The field can thus be found by equating the line integral of $\mathbf{E}_1$ on a circumference to the variation of the magnetic flux through the circle:

$$2\pi r E_1 = -\pi r^2 \partial_t B_0, \qquad E_1 = \frac{r}{2}\mu\omega n I_0 \sin\omega t. \tag{1}$$

This result can also be obtained by writing the equation $\boldsymbol{\nabla}\times\mathbf{E}_1 = -\partial_t \mathbf{B}_0$ in cylindrical coordinates and integrating for E_1 over r.

b) Since Ferrite is a conducting material, $\mathbf{E}_1$ drives a current density $\mathbf{J}_1 = \sigma\mathbf{E}_1$. The current density heats up the material due to Joule dissipation.The power loss by Joule heating per unit volume is

$$\mathbf{J}_1\cdot\mathbf{E}_1 = \sigma E_1^2 = \sigma(r^2/4)(\mu\omega n I_0 \sin\omega t)^2 \tag{2}$$

that has the average value over one cycle

$$\langle \mathbf{J}_1\cdot\mathbf{E}_1\rangle = \sigma E^2 = \sigma(r^2/8)(\mu\omega n I_0)^2 \tag{3}$$

being $\langle \sin^2\omega t\rangle = 1/2$. The total power can be found by integrating over the volume of the Ferrite cylinder:

$$P_d = \frac{\sigma}{8}(\mu n\omega I_0)^2 \int_0^a r^2 2\pi\ell r\,dr = \pi a^2 \ell \frac{\sigma}{16}(\mu n\omega I_0 a)^2. \tag{4}$$

c) The induced current $\mathbf{J}_1$ generates a magnetic field $\mathbf{B}_1$ that locally lowers the total field with respect to the static value $\mathbf{B}_0$. To evaluate $\mathbf{B}_1$, we notice that the current flowing in $\hat{\boldsymbol{\phi}}$ direction between r and $r+dr$ is equivalent to the current of a solenoid having radius r and surface current $d\iota = J_1(r)dr$. Thus, this infinitesimal current layer generates the field $d\mathbf{H}_1 = dH_1\hat{\mathbf{z}} = d\iota = J_1(r)dr$ in the region $r' < r$, and a null field for $r' > r$ (in fact ι is equivalent to nI for a solenoid). The total field at r is thus given by the integral

$$H_1 = H_1(r,t) = \int_r^a J_1(r')dr' = \frac{1}{4}(a^2 - r^2)\sigma\omega n\mu I_0 \sin\omega t \tag{5}$$

and $B_1 = \mu H_1$. The *total* magnetic field is $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 = [\mu n I(t) + B_1(r,t)]\hat{\mathbf{z}}$. The correction to the "static" field is strongest on the axis (because the contributions of all the eddy currents add up there) and null at the border ($r = a$).

To check that the approximation is consistent we check that the the peak value of B_1, i.e. $\max(B_1) = (\mu\sigma\omega a^2/4)B_0$, is much smaller than B_0. This gives the condition

$$\mu\sigma\omega a^2 = \mu_r\mu_0\epsilon_0\frac{\sigma}{\epsilon_0}\omega a^2 = \frac{\mu_r\omega a^2}{\tau c^2} \ll 1, \tag{6}$$

where $\tau = \epsilon_0/\sigma$ is the relaxation time. Thus, if a material with high μ_r is used, it must have a low conductivity to minimize the effect of eddy currents. Typical values for Ferrite are $\mu_r \sim 10^3$ and $\sigma \sim 5\ \Omega^{-1}\text{m}^{-1}$. At the frequency $f = \omega/2\pi = 50$ Hz we obtain from (6)

$$\mu\sigma\omega a^2 \sim 6 \times 10^{-2}(a/\text{m})^2. \tag{7}$$

Thus, the radius a of the solenoid should be kept small enough.

It is also instructive to compare the energy dissipated per cycle $U_d = (2\pi/\omega)P_d$ with the magnetic energy stored in the solenoid, $U_m = (\langle B_0^2/2\mu\rangle)\pi a^2\ell$. We obtain

$$\frac{U_d}{U_m} \simeq \frac{\pi}{4}\mu\sigma\omega a^2. \tag{8}$$

Thus, the condition (6) is also equivalent to the requirement that the energy loss to heating per unit time is small with respect to the stored magnetic energy.