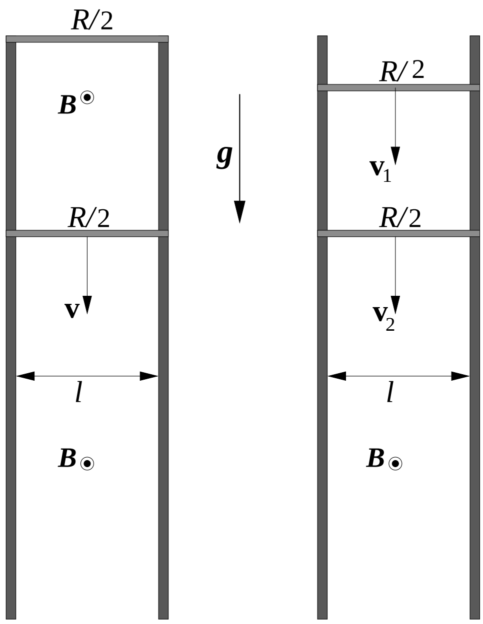
A circuit with "free-falling" parts

A circuit of rectangular shape and resistance R is place vertically in the gravity field $\mathbf{g}$. The two horizontal sides have length ℓ, mass M and can move freely in vertical direction keeping the electrical contact. The system is placed in an uniform and constant magnetic field $\mathbf{B}$ perpendicular to the plane of the circuit. (The variation of the resistance with the position of the moving sides and the self-induction of the circuit are assumed to be negligible).



Firstly, assume that the upper side is kept fixed, while the lower side starts its motion at $t = 0$. Let $v = v(t)$ its velocity at the time t, and $v(0) = 0$.

a) Find the equation of motion for $v(t)$ and give its solution showing that asymptotically the velocity assumes a constant value v_d.

b) When $v(t) = v_d$ find the power dissipated in the circuit due to Joule heating and the mechanical power due to the gravity force.

Now consider the case in which both moving sides are freed for $t > 0$, with the initial conditions $v_1(0) = v_0 \neq 0$ and $v_2(0) = 0$.

c) Find the equations of motion for the velocities of the two mobile sides $v_1(t)$ and $v_2(t)$ and its solution. Discuss the asymptotic behavior of $v_1(t)$, $v_2(t)$ and of the current in the circuit $I(t)$.

Solution

a) Let us fix the y axis oriented upwards and with the position of the upper side at $y = 0$. The current I in the circuit is given by

$$RI = \mathcal{E} = -\frac{d\Phi(\mathbf{B})}{dt} = -B\ell\frac{d(-y)}{dt} = +B\ell v. \tag{1}$$

The velocity has negative sign and the current circulates clockwise, in agreement with Lenz's law. The force on the moving side is $\mathbf{F} = -B\ell I\hat{\mathbf{y}}$ and it is directed oppositely to $\mathbf{g}$. Thus

$$M\frac{dv}{dt} = -Mg - B\ell I = -Mg - \frac{(B\ell)^2}{R}v. \tag{2}$$

The solution of (2) with $v(0) = 0$ is

$$v(t) = v_d\left(1 - e^{-t/\tau}\right), \qquad \tau = \frac{MR}{(B\ell)^2}, \qquad v_d = -g\tau = -\frac{MRg}{(B\ell)^2}. \tag{3}$$

For $t \to \infty$, the side falls with constant drift velocity v_d.
b) When $v(t) = v_d$ the power dissipated by Joule heating is

$$P_d = RI^2 = \frac{(B\ell v)^2}{R} = \left(\frac{Mg}{B\ell}\right)^2 R. \tag{4}$$

The mechanical power is

$$P_m = M\mathbf{g}\cdot\mathbf{v} = Mg\frac{MgR}{(B\ell)^2} = P_d, \tag{5}$$

in agreement with energy conservation.
c) The current is now given by

$$RI = \mathcal{E} = -\frac{d\Phi(\mathbf{B})}{dt} = -B\ell\frac{d}{dt}(y_1 - y_2) = +B\ell(v_2 - v_1). \tag{6}$$

On the two mobile sides we have $\mathbf{F}_1 = B\ell I\hat{\mathbf{y}}$ and $\mathbf{F}_2 = -B\ell I\hat{\mathbf{y}}$, respectively. The equations of motion are

$$M\frac{dv_1}{dt} = -Mg + \frac{(B\ell)^2}{R}(v_2 - v_1), \qquad M\frac{dv_2}{dt} = -Mg - \frac{(B\ell)^2}{R}(v_2 - v_1). \tag{7}$$

By taking the sum and the difference of (7) we obtain

$$\frac{d}{dt}(v_2 - v_1) = -\frac{1}{\tau}(v_2 - v_1), \qquad \frac{d}{dt}(v_2 + v_1) = -2g, \tag{8}$$

which have the solutions

$$v_2 - v_1 = v_0 e^{-t/\tau}, \qquad v_2 + v_1 = v_0 - 2gt, \tag{9}$$

from which we obtain:

$$v_{1,2} = \frac{v_0}{2}(1 \mp e^{-t/\tau}) - gt. \tag{10}$$

For $t \gg \tau$

$$v_1 \simeq v_2 \simeq \frac{v_0}{2} - gt. \tag{11}$$

The current $I(t) = 0$ as $v_1 = v_2$.