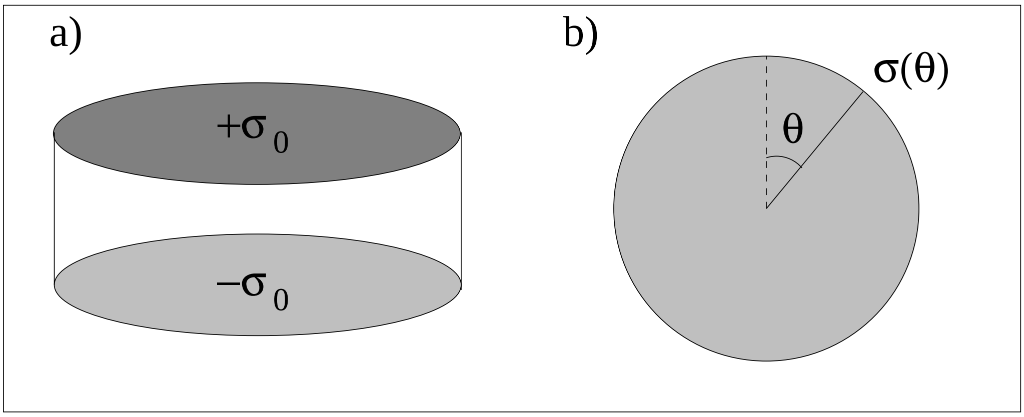
Geometry effects on charge relaxation

A cylinder of radius a and height $h \ll a$ is made of a conducting material of resistivity ρ. At $t = 0$, opposite surface charge densities $\pm\sigma$ are uniformly distributed on the two bases [see figure a)]. Boundary effects are negligible.



a) Find the temporal variation of the charge densities at $t > 0$.
b) Determine the energy dissipated in the cylinder.
c) Discuss whether the flow of a current through the cylinder leads to the generation of a magnetic field.
Now consider a sphere of radius b and resistivity ρ, with a surface density distribution at $t = 0$ given by $\sigma(\theta) = \sigma_0 \cos\theta$ [see figure b)].
d) Give answers **a)-c)** for the spherical conductor.

Solution

a) An uniform surface charge density σ generates an uniform electric field of modulus $E = \sigma/\epsilon_0$ and perpendicular to the base surfaces (all vector quantities are parallel to $\mathbf{E}$ thus we can drop the vector signs for briefity). Inside the cylinder the current density $J = E/\rho$. The continuity equation gives $\partial_t \sigma = -J$. Combining these relations we obtain

$$\partial_t \sigma = -\frac{1}{\rho\epsilon_0}\sigma, \qquad \sigma(t) = \sigma_0 e^{-t/\tau}, \qquad \tau = \rho\epsilon_0. \tag{1}$$

b) The power dissipated by Joule heating is given by

$$P_d = \int_{\text{cyl}} JEd^3x = (E_0^2/\rho)e^{-2t/\tau}(\pi a^2 h), \tag{2}$$

where $E_0 = E(t=0) = \sigma_0/\epsilon_0$. Thus, the total dissipated energy is

$$U_d = \int_0^\infty P_d dt = (E_0^2/\rho)(\tau/2)(\pi a^2 h) = (\epsilon_0 E_0^2/2)(\pi a^2 h) \tag{3}$$

that is equal to the electrostatic energy at $t = 0$.

c) The source term for the magnetic field is

$$\mu_0(J + \epsilon_0 \partial_t E) = \mu_0 E/\rho - \mu_0(\epsilon_0/\tau)E = \mu_0(E/\rho - E/\rho) = 0 \tag{4}$$

thus no magnetic field is generated.

d) A surface charge distribution $\sigma = \sigma_c \cos\theta$, where σ_c may depend on time, produces inside the sphere an uniform field $E = \sigma_c/3\epsilon_0$ directed along the z axis ($\theta = 0$). The current density is given by $\mathbf{J} = \mathbf{E}/\rho$ and the continuity equation gives at any point of the surface

$$\partial_t(\sigma_c \cos\theta) = -\mathbf{J}\cdot\mathbf{n} = -J\cos\theta, \tag{5}$$

from which $\partial_t \sigma_c = -J$ holds. (It is thus self-consistent to assume that the spatial distribution of the charge density, the current and the field is the same at any t). We obtain that all fields decay as $\exp(-t/\tau_s)$ where now $\tau_s = 3\epsilon_0\rho$.

The dissipated energy is again given by the integral of $\mathbf{J}\cdot\mathbf{E}$ over the volume of the sphere and over time, that yields

$$U_d = (E_0^2/\rho)(\tau_s/2)(4\pi b^2/3) = 3(\epsilon_0 E_0^2/2)(4\pi b^2/3) \tag{6}$$

where $E_0 = E(t=0) = \sigma_0/3\epsilon_0$. It can be easily shown that U_d is equal to the electrostatic energy at $t = 0$. (In performing the integral of $\epsilon_0 E^2/2$ over the whole space to find the electrostatic energy, the "outer" region where the field is that of a dipole gives a contribution twice that of the "inner", i.e. sphere region where the field is uniform).

The source term for the magnetic field does not vanish anymore

$$\mu_0(J + \epsilon_0 \partial_t E) = \mu_0 E/\rho - \mu_0(\epsilon_0/\tau_s)E = \mu_0(E/\rho - E/3\rho) = (2\mu_0/3)E \tag{7}$$

and thus $\mathbf{B} \neq 0$. For symmetry reasons, $\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}}$ and by calculating its line integral along a field line we obtain

$$2\pi r B_\phi = (2\mu_0/3)E(\pi r^2), \qquad B_\phi = \mu_0 E_0 (r/3) e^{-t/\tau_s}. \tag{8}$$