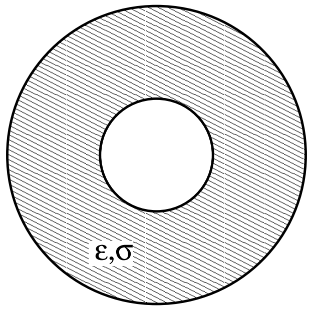
Charge decay in a lossy spherical capacitor



A spherical capacitor is delimited by two concentrical, spherical metallic electrodes, having radii a and $b > a$ respectively. The inner space is filled with a "non-perfect dielectric" material having dielectric permittivity ϵ and conductivity σ. At $t = 0$, the charge on the inner surface is Q.

a) Find the decay time of the capacitor.

b) Find the power dissipated by Joule heating inside the capacitor and compare it with the temporal variation of the electrostatic energy.

Solution

a) The system has a spherical symmetry, thus the electric field will be in the radial direction and, apart from the temporal variation, will depend only on the distance r from the centre of the capacitor.

Let $q = q(r,t)$ be the charge contained at the time t inside the spere of radius r. From Gauss's theorem, the electric field at a distance r and at the time t is given by

$$E(r,t) = \frac{1}{4\pi\epsilon}\frac{q(r,t)}{r^2} \qquad (a < r < b)\,.$$

(For $r < a$, obviously $E(r,t) = 0$, and the same holds for $r > b$ if the total charge on the capacitor is zero).

The current density is given by $\mathbf{J}(r,t) = \sigma\mathbf{E}(r,t)$. Due to the conservation of the charge, the flux of $\mathbf{J}$ on the surface of the sphere of radius r is opposite to the temporal variation of the charge inside the sphere:

$$\frac{dq}{dt} = -\Phi(\mathbf{J}) = -4\pi r^2 J = -4\pi r^2 \sigma E = -\frac{\sigma}{\epsilon}q\,,$$

from which, defining $\tau = \epsilon/\sigma$, we obtain

$$q(r,t) = q(r,0)e^{-t/\tau} = Qe^{-t/\tau}\,.$$

Note that the free charge density at any time is located only on the surface of the conductors, and a volume charge density never appears. In fact, using the continuity equation

$$\partial_t \rho = -\boldsymbol{\nabla}\cdot\mathbf{J} = -\sigma\boldsymbol{\nabla}\cdot\mathbf{E} = 0\,,$$

since $\mathbf{E}$ is the field of a point charge and the material is uniform ($\nabla\sigma = 0$).

b) The power dissipated over the volume of the capacitor is given by

$$P_d = \int \mathbf{J}\cdot\mathbf{E}dV = \int \sigma E^2(r,t)dV = \sigma e^{-2t/\tau}\int E^2(r,0)dV\,.$$

The electrostatic energy is

$$U_{es} = \int \frac{\epsilon}{2}E^2(r,t)dV = \frac{\epsilon}{2\sigma}P_d\,,$$

thus, being $\tau = \sigma/\epsilon$,

$$\frac{dU_{es}}{dt} = -\frac{2}{\tau}U_{es} = -P_d\,.$$

The temporal variation of the electrostatic energy is equal to the power dissipated by Joule heating.

Notice that the results do not depend on b (the outer radius of the capacitor) thus they would be the same for a sphere "suddenly" immersed in an infinite conducting medium.