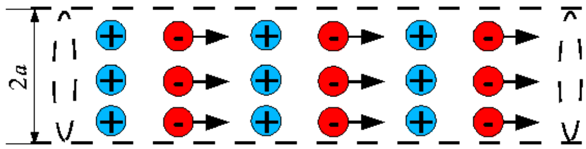
Pinch effect



An uniformly distributed current **J** flows along an infinite cylindrical conductor of radius a. The current is due to electrons having number density n_e and constant drift speed v_e (parallel to the axis of the cylinder). Ions are considered to be fixed and having an uniform density n_i and charge Ze. The system is globally neutral.

a) Find the magnetic field generated by the current and the related magnetic force on the electrons.

The electron displacement due to the magnetic force leads to the generation of a back-holding electric field, which keeps electrons in mechanical equilibrium.

b) Find the electric field generated in order to compensate the magnetic force on the electrons, and the related charge distribution.

c) Evaluate the effect in "standard" conditions for a good Ohmic conductor.

Solution

a) Let us fix a cylindrical coordinate frame with the axis z along the axis of the cylinder. Assume $v_e > 0$, i.e. the electrons flow in the positive direction of the z axis. Using Ampère's theorem, we find an azimuthal magnetic field $B_\phi = \mu_0 Jr/2$ where $J = -en_e v_e < 0$. Thus $B_\phi < 0$, i.e. it is oriented clockwise with respect to the z axis. The magnetic force $\mathbf{F} = -e(\mathbf{v} \times \mathbf{B})$ is in the radial direction and it is directed towards the axis:

$$\mathbf{F} = -(\mu_0 e^2 n_e v_e^2/2)\mathbf{r}.$$

Thus, the magnetic force tends to push the electrons in the inward direction.
b) The Lorentz force is $\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Thus, we must have $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ in order that $\mathbf{F} = 0$. Thus

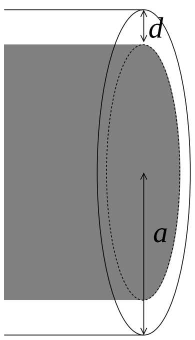
$$\mathbf{E} = -(\mu_0 e n_e v_e^2/2)\mathbf{r}.$$

From Gauss's theorem we find that this field is generated by a charge density ρ which is uniform over the cylinder, such that $E = \rho r/2\epsilon_0$. Thus $\rho = -\epsilon_0\mu_0 e n_e v^2 = -en_e(v/c)^2$.

Since $\rho = e(Zn_i - n_e)$ also holds, we obtain

$$n_e = \frac{Zn_i}{1 - v^2/c^2}.$$

The electron density is uniform, but it exceeds the value $n_{e0} = Zn_i$ that would ensure $\rho = 0$. The "missing" positive charge is located on the surface of the conductor.
c) For an electron in an Ohmic conductor usually $v \approx 1$ cm/sec $= 10^{-2}$ m/sec, thus $(v/c)^2 \approx 10^{-21}$.



To get convinced that the effect is negligibly small in a standard conductor, let us assume that the piling up of the electrons in a central region of radius $a - d$ of a wire of radius a leaves a layer of ion density between $r = a - d$ and $r = a$. The thickness d can be estimated by the constraint of charge conservation, i.e. the ion charge δQ inside the outer layer, must compensate the excess electron charge in the center. By considering the charge balance for a length ℓ we have

$$\delta Q = Zen_i\pi[a^2 - (a-d)^2]\ell \doteq -\rho\pi(a-d)^2\ell, \tag{1}$$

which to order $(v/c)^2 \ll 1$ and assuming consistently $d \ll a$, can be written as

$$Zen_i(2\pi ad) \simeq -\rho(\pi a^2) \simeq Zen_i(v/c)^2(\pi a^2). \tag{2}$$

We thus get $d \simeq (a/2)(v/c)^2$, so that making d of the order of the crystal lattice spacing ($\simeq 10^{-10}$ m) would require $a \simeq 10^{11}$ m, a remarkably large radius!

One may also observe that the presence of a surface charge density $\sigma \sim en_i d \sim en_i a(v/c)^2$ (we just keep the order of magnitude) implies an electrostatic pressure $P_{\text{es}} = \sigma^2/2\epsilon_0$ directed radially inwards. This pressure must be balanced by the elastic deformation of the wire, that produces a pressure

$$P_{\text{el}} \sim Y\frac{\delta a}{a} \tag{3}$$

where δa is the variation of the wire radius due to the electrostatic stress, and Y is the Young modulus. Since $Y \sim 10^{11}$ N m^{-2}, it is easy to see that if $a \sim 1$ cm one would get $\delta a \sim 10^{-14} a$, that is of the order of the nuclear radius!

The pinch effect is important indeed in discharges where very large currents flow and the electrons are very energetic, such as columns of hot plasma (originally studied as devices for controlled thermonuclear fusion; here $v \sim 0.1 \sim 0.01c$ may hold) or even lighthings. The electrostatic pressure generated by the electron displacement makes the ions collapse, shrinking the column. However, the electrostatic pressure can be in principle balanced by the thermal pressure of the column.