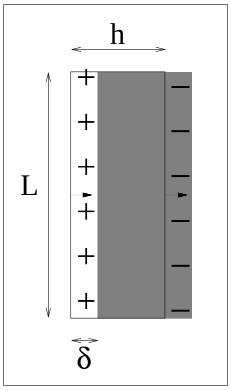
Plasma oscillations

A metallic slab has a square base of side L and a thickness $h \ll L$. The electron and ion density in the slab are $n_e = n$ and $n_i = n_e/Z$, where Z is the ion charge.

Through the appliance of an external field, all the electrons are displaced by $\delta < h$ in the direction perpendicular to the slab surface. We assume that the densities of both electrons and ions are constant, that ions are immobile and that edge effects are negligible.



a) Find the electrostatic field generated by the displacement of electrons.
b) Find the electrostatic energy.
The external force is removed, and the "electron slab" starts to oscillate around its equilibrium position.
c) Find the oscillation frequency, in the small displacement limit ($\delta \ll h$).

Solution

a) Due to the electron rigid displacement, the charge density has the spatial distribution

$$\rho(x) = \begin{cases} +en & (0 < x < \delta)\,, \\ 0 & (\delta < x < h\,,\; x < 0\,,\; x > h+\delta)\,, \\ -en & (h < x < h+\delta)\,. \end{cases}$$

From $\rho(x)$ we obtain the electrostatic field $\mathbf{E}$ by integrating the equation $\boldsymbol{\nabla}\cdot\mathbf{E} = \partial_x E_x = \rho/\epsilon_0$:

$$E_x(x) = \frac{en}{\epsilon_0}\begin{cases} x & (0 < x < \delta)\,, \\ \delta & (\delta < x < h)\,, \\ h+\delta-x & (h < x < h+\delta)\,, \\ 0 & (x < 0\,,\; x > h+\delta)\,. \end{cases}$$

b) To find the electrostatic energy we integrate the "energy density" $u = (\epsilon_0/2)E_x^2$ over the whole volume:

$$\begin{aligned} U_{es} &= \int \frac{\epsilon_0}{2}E_x^2 dV = \frac{\epsilon_0}{2}L^2\int_0^{h+\delta} E_x^2 dx \\ &= \frac{\epsilon_0}{2}L^2\left(\frac{en}{\epsilon_0}\right)^2\left(\int_0^\delta x^2 dx + \int_\delta^h \delta^2 dx + \int_h^{h+\delta}(h+\delta-x)^2 dx\right) \\ &= \frac{(enL)^2}{2\epsilon_0}\left[\frac{\delta^3}{3} + \delta^2(h-\delta) + \frac{\delta^3}{3}\right] = \frac{(enL)^2}{2\epsilon_0}\left(h\delta^2 - \frac{\delta^3}{3}\right)\,. \end{aligned}$$

We used $dV = L^2 dx$.

c) In the limit $\delta \ll h$ we have $U_{es} \simeq (enL)^2 h\delta^2/2\epsilon_0$, which is the potential energy of an harmonic oscillator. Note that the expression of U_{es} remains the same also when the electrons move in the $x < 0$ region.

The force on the electron slab is

$$F = -\frac{\partial U_{es}}{\partial \delta} = -\frac{(enL)^2 h\delta}{\epsilon_0}\,.$$

The equation of motion is

$$M\ddot{\delta} = F \equiv -M\omega^2\delta\,,$$

where $M = m_e n L^2 h$ is the total mass of the electron slab. We thus find

$$\omega^2 = \frac{ne^2}{\epsilon_0 m_e} \equiv \omega_p^2\,,$$

where the *plasma frequency* ω_p is a typical parameter of any conductor, which depends only on the density of (free) electrons.