Charge relaxation in a conducting sphere

A conducting sphere of radius a and Ohmic conductivity σ has a net charge Q. A the time $t = 0$ the charge is uniformly distributed over the volume of the sphere, with density $\rho_0 = Q/(4\pi a^3/3)$. Since in steady conditions the charge in a isolated conductor can only be located on the conductor's surface, for $t > 0$ the charge must progressively migrate towards the surface of the sphere.
a) Find the temporal law and the characteristic time τ of the evolution of the free charge density inside the sphere for $t > 0$. Evaluate τ for a good conductor (e.g. Copper).
b) Find the variation of the electrostatic energy during the redistribution of the charge.
c) Show that the energy dissipated due to Joule heating equals the variation of the electrostatic energy.

Solution

a) Let $q(r,t)$ be the charge contained inside the spherical surface of radius r at the time t (with $r < a$, $t > 0$). For symmetry reasons the electric field is in the radial direction and, applying Gauss's theorem, is given by

$$E(r,t) = \frac{1}{4\pi\epsilon_0}\frac{q(r,t)}{r^2}. \tag{1}$$

Due to continuity equation the flux of the current density $\mathbf{J} = \sigma\mathbf{E}$ through the spherical surface is equal to the temporal variation of $q(r,t)$:

$$4\pi r^2 J(r,t) = 4\pi r^2 \sigma E(r,t) = -\partial_t q(r,t). \tag{2}$$

By eliminating E from (1-2) we obtain the equation

$$\partial_t q(r,t) = -\frac{\sigma}{\epsilon_0} q(r,t), \tag{3}$$

that has the solution

$$q(r,t) = q(r,0)e^{-t/\tau}, \qquad \tau \equiv \frac{\epsilon_0}{\sigma}. \tag{4}$$

Since at $t = 0$ the charge density is uniform ($q(r,0) = Q(r/a)^3$), it remains uniform for $t > 0$: $\rho(t) = \rho_0 e^{-t/\tau}$.

Using the continuity equation, we can also obtain the variation of the *surface* charge density $\Sigma = \Sigma(t)$:

$$\partial_t \Sigma = +J(a,t) = \sigma E(a,t) = \frac{Q}{4\pi a^2 \tau} e^{-t/\tau}, \tag{5}$$

so that, asymptotically,

$$\Sigma(\infty) = \int_0^\infty \partial_t \Sigma dt = \frac{Q}{4\pi a^2 \tau}\int_0^\infty e^{-t/\tau} dt = \frac{Q}{4\pi a^2}. \tag{6}$$

The charge relaxation time $\tau = \epsilon_0/\sigma$ is extremely short in a good conductor. For Copper, $\sigma \simeq 10^7\ \Omega^{-1}\text{m}^{-1}$, thus

$$\frac{\epsilon_0}{\sigma} \simeq \frac{8.854 \times 10^{-12}}{10^7}\ \text{s} \sim 10^{-18}\ \text{s} = 1\ \text{as} \tag{7}$$

(1 as= 1 *attosecond*).

b) To compute quickly the variation of electrostatic energy ΔU_{es} we notice that the electric field outside the conducting sphere ($r > a$) is constant, while inside the sphere it decays from the initial profile $E(r,0) = (Q/4\pi\epsilon_0)(r/a^3)$ to $E(r,\infty) = 0$. Thus, using the "energy density" $u_{es} = \epsilon_0 E^2/2$ we can write

$$\Delta U_{es} = -\int_{\text{sfera}} \frac{\epsilon_0}{2} E^2(r,0) dV = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \int_0^a \frac{r^2}{a^6} 4\pi r^2 dr = \frac{1}{10}\frac{Q^2}{4\pi\epsilon_0 a}. \tag{8}$$

c) The temporal derivative of the electrostatic energy can be written as

$$\begin{aligned}
\partial_t U_{es} &= \partial_t \int_0^\infty \frac{\epsilon_0}{2} E^2(r,t) 4\pi r^2 dr = \int_0^a \frac{\epsilon_0}{2} \partial_t E^2(r,t) 4\pi r^2 dr = \int_0^a \frac{\epsilon_0}{2} \left(-\frac{2}{\tau}\right) E^2(r,t) 4\pi r^2 dr \\
&= -\frac{\epsilon_0}{\tau} \int_0^a E^2(r,t) 4\pi r^2 dr. \qquad (9)
\end{aligned}$$

We used $E^2(r,t) = E^2(r,0)e^{-2t/\tau}$ for $r < a$ and $E^2(r,t) = E^2(r,0)$ for $r > a$.

The power loss due to Joule heating is

$$P_d = \int_0^\infty \mathbf{J} \cdot \mathbf{E} 4\pi r^2 dr = \int_0^a \sigma E^2(r,t) 4\pi r^2 dr, \qquad (10)$$

being $\mathbf{J} = \sigma \mathbf{E}$ for $r < a$ and $\mathbf{J} = 0$ for $r > a$. Being $\epsilon_0/\tau = \sigma$, we obtain $P_d = -\partial_t U_{es}$.