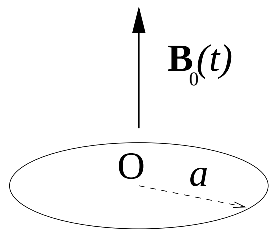
Rotation driven by induction



An homogeneous ring of radius a and negligible thickness has a mass M. A charge Q is uniformly distributed along the radius. The ring is placed on a plane over which it can rotate around its center O without friction.
At $t = 0$, an uniform, time varying magnetic field $\mathbf{B}_0(t)$, perpendicular to the ring's plane, is turned on.
a) Neglecting self-induction effects, find the angular velocity $\omega = \omega(t)$ of the ring as a function of $\mathbf{B}_0(t)$.
b) Find the magnetic field generated at O by the rotation of the ring.
c) Discuss how the result of a) is modified by self-induction effects.

Solution

a) Since the magnetic field varies in time, there will be an electric field $\mathbf{E}$ according to Faraday's law $\boldsymbol{\nabla} \times \mathbf{E} = -\partial_t \mathbf{B}_0$, from which we see that $\mathbf{E}$ lies on the plane of the ring.
Let us consider an infinitesimal element $d\mathbf{l}$ of the ring's circumference, over which the electric force is given by

$$d\mathbf{F} = \mathbf{E}dq = \mathbf{E}\frac{Q}{2\pi a}|d\mathbf{l}|. \tag{1}$$

The ring's rotation is driven by the force component which are tangential to the circumference, providing a total momentum

$$\mathbf{M} = \int_{\text{circ}} \mathbf{r} \times d\mathbf{F} = a \int_{\text{circ}} \hat{\mathbf{r}} \times \mathbf{E}dq = \hat{\mathbf{z}}\frac{Q}{2\pi}\int_{\text{circ}} \mathbf{E}\cdot d\mathbf{l} = -\hat{\mathbf{z}}\frac{Q}{2\pi}\frac{d\Phi(\mathbf{B}_0)}{dt} = -\frac{Qa^2}{2}\frac{d\mathbf{B}_0}{dt}. \tag{2}$$

The rotation of the ring is described by the equation

$$\mathcal{I}\frac{d\boldsymbol{\omega}}{dt} = \mathbf{M}, \tag{3}$$

where $\mathcal{I} = Ma^2$ is the momentum of inertia. Thus we obtain

$$\frac{d\boldsymbol{\omega}}{dt} = -\frac{Qa^2}{2\mathcal{I}}\frac{d\mathbf{B}_0}{dt}, \qquad \boldsymbol{\omega}(t) = -\frac{Q}{2M}\mathbf{B}_0(t). \tag{4}$$

At any time the angular velocity is proportional to the magnetic field.
b) The rotating ring is analogous to a circular coil with the current $I = Q/T = \omega Q/2\pi$. Thus, the value of the induced field at the center of the ring is given by $B_i = \mu_0 I/2a$. From the result of point a) we obtain that the induced field is in the direction opposite to $\mathbf{B}_0$, in agreement with Lenz's law:

$$\mathbf{B}_i = -\frac{\mu_0 Q^2}{8\pi Ma}\mathbf{B}_0. \tag{5}$$

c) Due to the self-induction effects we expect the "electromotive force" $\mathcal{E} = \int_{\text{circ}} \mathbf{E}\cdot d\mathbf{l}$ computed in a) to be lower by an amount $LdI/dt = L(Q/2\pi)d\omega/dt$, where L is the self-inductance of the equivalent coil. Thus, the equation of motion is modified as follows

$$\frac{d\omega}{dt} = -\frac{Q}{2M}\frac{dB_0}{dt} - \frac{Q^2 L}{4\pi^2 Ma^2}\frac{d\omega}{dt}, \tag{6}$$

which we may rewrite as

$$\left(1 + \frac{Q^2 L}{4\pi^2 Ma^2}\right)\frac{d\omega}{dt} = -\frac{Q}{2M}\frac{dB_0}{dt}. \tag{7}$$

The effect of self-induction is thus equivalent to an increase of the ring's inertia, as if the ring would have an "effective mass" $M' = M(1 + Q^2L/4\pi^2Ma^2) > M$; with respect to what found in a), at the same time the angular velocity is lower by the factor M/M'.