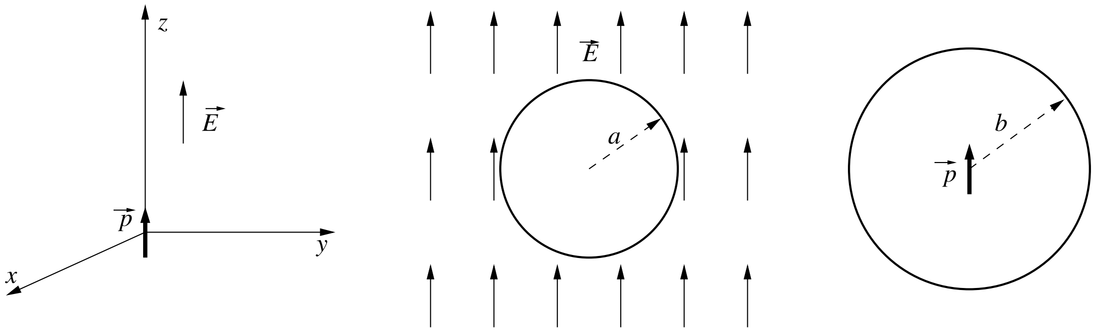
A solution seeking for a problem

An electric dipole $\mathbf{p}$ is located in the origin of a cartesian frame, in the presence of an uniform electric field $\mathbf{E}$, parallel to $\mathbf{p}$.



a) b) c)

a) Find the total electrostatic potential $V = V(\mathbf{r})$, showing that the equipotential surface $V = 0$ is a sphere and calculate the radius R of the latter.
Now use the result from point **a)** to find the electric potential in the whole space for the following problems:
b) A conducting sphere of radius a is placed in an uniform electric field $\mathbf{E}_0$;
c) a dipole $\mathbf{p}_0$ is placed in the center of a conducting spherical shell of radius b.
d) Find the solution to problem **c)** using the method of image charges.

Solution

a) Let us assume that $\mathbf{p}$ and $\mathbf{E}$ are along the z axis. The electric potential is the sum of the potential of the dipole and of the potential corresponding to an uniform electric field,

$$V(\mathbf{x}) = k_0 \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} - Ez = k_0 \frac{p\cos\theta}{r^2} - Er\cos\theta\,,$$

where $k_0 = 1/4\pi\epsilon_0$ and θ is the angle between $\mathbf{r}$ and the z axis. The surface at $V = 0$ is defined by

$$k_0 \frac{p\cos\theta}{r^2} - Er\cos\theta = 0\,,$$

and is therefore a sphere of radius

$$r = \left(\frac{k_0 p}{E}\right)^{1/3} \equiv R\,. \tag{1}$$

b) We must find a solution for the potential V which satisfies the condition $V = 0$ at the surface of the conductor, i.e. $V(|\mathbf{r}| = a) = 0$, and such that at large distance from the conductor the field is $\mathbf{E}_0$.
From point **a)** we obtain that such a solution is constructed by assuming that the field generated by the surface charges on the sphere is the field of a dipole $\mathbf{p}_i$, located in the center of the sphere and parallel to $\mathbf{E}_0$, which module is defined by Eq.(1) for $E = E_0$ and $R = a$. We thus find

$$\mathbf{p}_i = \frac{a^3}{k_0}\mathbf{E}_0 = 4\pi\epsilon_0 a^3 \mathbf{E}_0 = 3\epsilon_0 \mathcal{V}_a \mathbf{E}_0\,,$$

where $\mathcal{V}_a$ is the volume of the sphere. The potential for $r > a$ is thus given by

$$V = k_0 \frac{\mathbf{p}_i\cdot\mathbf{r}}{r^3} - E_0 z\,, \tag{2}$$

while $V = 0$ for $r < a$. Notice that the total charge induced on the sphere is zero, thus the solution is the same either for a grounded or an isolated, uncharged sphere.
c) In this case the boundary condition $V = 0$ is at $r = b$. The polarization charges on the inner surface generate an uniform field $\mathbf{E}_i$ parallel to $\mathbf{p}_0$ and of module given by Eq.(1) for $p = p_0$ and $r = b$:

$$\mathbf{E}_i = \frac{k_0}{b^3}\mathbf{p}_0 = \frac{\mathbf{p}_0}{3\epsilon_0 \mathcal{V}_b} = \frac{\mathbf{p}_0}{4\pi\epsilon_0 b^3}\,.$$

As in the preceding case, the total induced charge is zero and thus it does not matter whether the shell is grounded or isolated and uncharged.
d) We model the dipole as two point charges $\pm q$ located at $z = \pm d/2$. From the already known problem of a point charge inside a conducting shell, we obtain that a charge Q placed at a distancd z from the center of the shell induces a surface charge distribution equivalent to an image charge $Q' = -Qb/z$ located at $z' = b^2/z$. The field inside the shell is thus that due to the two real charges plus that due to two image charges $\mp 2qb/d$ located at $\pm z = \pm 2b^2/d$, respectively.

Taking the limit $d \to 0$ but $qd \to p$, the field of the real charges becomes exactly that of a dipole $\mathbf{p} = p\hat{\mathbf{z}}$ located in the center of the shell, while the field of the image charges is uniform. Let us take the field of the image charges at the center of the shell:

$$E_c = 2 \times k_0 \frac{2qb/d}{(2b^2/d)^2} = k_0 \frac{qd}{b^3} = k_0 \frac{p}{b^3} \,,$$

which coincides with the value found at point **c)**. Of course, a similar method may be used also to recover the result of point **b)**.