Collision of two charged spheres

Two undeformable spheres have both radius R and mass M and opposite charges $\pm Q$, uniformly distributed in their volume. The two spheres attract each other, starting at rest from an infinite distance.
a) Find the initial energy of the system.
b) Find the velocity of the spheres when they touch each other (i.e. when the distance between the centers is $d = 2R$).
After the contact, the two spheres keep moving and interpenetrate each other, without any friction.
c) Show that the field in the region where the two spheres overlap is uniform and find its value as a function of the distance d ($< 2R$) between the centers.
d) Find the velocity of the spheres when the two centers overlap ($d = 0$).

Solution

a) When the distance d between the centers of the two spheres is larger than $2R$ (no overlap of charge distributions), the potential energy U of the system can be written as

$$U = 2U_0 + U_{\text{int}}$$

where U_0 is the electrostatic energy stored in a single sphere and U_{int} is the energy due to the mutual interaction between the spheres.

One of the possible ways to find U_0 is to imagine to build up the sphere adding subsequent infinitesimal layers of charge (carried from infinite distance). From Gauss's theorem we know that, for an uniformly charged sphere having charge density ρ, radius r, and total charge $q = q(r) = \rho(4\pi r^3/3)$, the field and the potential outside the sphere are those of a point charge q located in the center. Thus, in building the sphere, when a new layer of charge $dq = \rho 4\pi r^2 dr$ is added, its charge will be located at the potential $V(r) = k_0 q(r)/r$, where $k_0 = 1/4\pi\epsilon_0$; thus, the corresponding stored amount of electrostatic energy is $V(r)dq$. Integrating over r up to the final radius R we find

$$U_0 = \int_0^R k_0 \left(\frac{4\pi}{3} r^3\right) \rho \frac{1}{r} \rho 4\pi r^2 dr = \frac{4\pi\rho^2}{3\epsilon_0} \int_0^R r^4 dr = \frac{4\pi\rho^2 R^5}{15\epsilon_0} = \frac{3}{5} k_0 \frac{Q^2}{R},$$

being $\rho = Q/(4\pi R^3/3)$.

To find U_{int}, we note that if the distance $d > 2R$ the force between the spheres is identical to the force between two point charges $\pm Q$ placed in the cneters of the spheres. Thus

$$U_{\text{int}}(d) = -k_0 \frac{Q^2}{d}.$$

The total electrostatic energy is $U_{\text{tot}} = 2U_0 + U_{\text{int}}$. The initial energy is given by $2U_0$ because $U\text{int} = 0$ for $d \to \infty$.

b) Both the total momentum and energy of the system are conserved. Thus, the velocities of the two sphere are always equal and opposite. When $d = 2R$ the total kinetic energy is

$$2\frac{1}{2}Mv^2 = 2U_0 + U_{int}(2R) = k_0 \frac{Q^2}{2R},$$

and thus we obtain

$$v = \sqrt{k_0 \frac{Q^2}{2MR}}.$$

c) The electric field inside an uniformly charged sphere is

$$\mathbf{E}(\mathbf{r}) = \frac{\rho}{3\epsilon_0}\mathbf{r} = k_0 \frac{Q}{R^3}\mathbf{r},$$

and thus, at any point in the region where the spheres overlap, we obtain for the total field

$$\mathbf{E} = k_0 \frac{Q}{R^3}\mathbf{r}_1 + k_0 \frac{-Q}{R^3}\mathbf{r}_2 = k_0 \frac{Q}{R^3}(\mathbf{r}_1 - \mathbf{r}_2) = k_0 \frac{Q}{R^3}\mathbf{d}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the distances of the point from the center of the positive and of the negative sphere, respectively, and $\mathbf{d}$ is the distance between the two centers. Thus the field is uniform and parallel to $\mathbf{d}$.

d) When the centers of the two spehres coincide, the charge density and the electrostatic field are zero. The electrostatic energy is zero and thus

$$2\frac{1}{2}Mv^2 = U(\infty) = 2U_0,$$

$$v = \sqrt{\frac{6k_0}{5}\frac{Q^2}{MR}}.$$