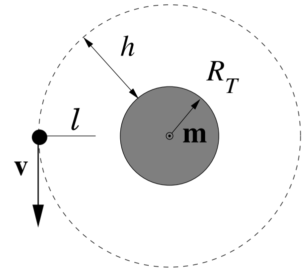
The Tethered satellite system

The Earth's magnetic field can be roughly approximated by the field of a dipole placed at the Earth's centre. The peak value is about 7.5×10^{-5} T.

a) Knowing that the Earth's radius is $R_T = 6400$ km, estimate the value of the Earth's magnetic dipole.

A satellite moves on the magnetic equatorial plane with a velocity $v \simeq 8$ km/sec at a constant height $h \simeq 100$ km over the Earth's surface, as shown in the figure (not to scale!).



A metallic wire of length $\ell = 1$ km hangs from the satellite, along the radial direction.

b) Find the electromotive force along the wire.

c) The satellite travels through the ionosphere, where charge carriers in outer space are available to close the circuit, thus a current can flow along the wire. Find the power dissipated by Joule heating in the wire and the mechanical force on the wire as a function of its resistance R.

Solution

a) The magnetic field of a dipole can be written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}[3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}} - \mathbf{m})],$$

and its modulus is proportional to

$$B^2 \propto [3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}} - \mathbf{m})]^2 = (3\cos^2\theta + 1)m^2,$$

being θ the angle between $\mathbf{m}$ and $\mathbf{r}$. The peak value of the magnetic field of on a sphere of radius r thus occurs for $\cos\theta = 1$, i.e. along the dipole axis. Thus, the maximum value of the Earth's magnetic field is related to m and R_T as follows:

$$B_{max} = \frac{\mu_0 m}{2\pi R_T^3}.$$

From the knowledge of B_{max} we obtain

$$m = \frac{2\pi}{\mu_0} R_T^3 B_{max} = 0.98 \times 10^{14} \text{ SI unis.}$$

b) In the equatorial plane ($z = 0$), the field is along z and it is given by

$$B_{eq}(r) = -\frac{\mu_0 m}{4\pi r^2}.$$

The EMF is given by the line integral of the magnetic force,

$$\mathcal{E} = \int_{\text{filo}} d\mathbf{l}\cdot\mathbf{v}(r)\times\mathbf{B}(r) = \int_h^{h-\ell} dr\omega r B_{eq}(r),$$

where $\omega = v/(R_T + h)$ is the angular velocity of the satellite. The integral is straightforward, however, noting that $\ell \ll h \ll R_T$, it is already a good approximation to assume $v(r)$ and $B_{eq}(r)$ as uniform along the wire:

$$\mathcal{E} \simeq v\ell B_0(R_T) = 300 \text{ Volts},$$

being $B_0(R_T) = B_{max}/2$.

c) The current and the dissipated power are

$$I = \frac{\mathcal{E}}{R}, \qquad P_d = \frac{\mathcal{E}^2}{R},$$

and the mechanical force is

$$\mathbf{F} = I\ell\hat{\mathbf{r}}\times\mathbf{B}_{eq} = -\frac{B_{eq}^2\ell^2}{R}\mathbf{v}, \qquad \mathbf{F}\cdot\mathbf{v} = -P_d.$$

We obtain

$$F \simeq 11 \text{ N} R_\Omega^{-1},$$

where R_Ω is the resistance in Ohms. If we consider a Copper wire ($\sigma \simeq 10^7\ \Omega^{-1}\text{m}^{-1}$) of 1 cm^2 section, R is of the order of 1 Ohm for $\ell = 1$ km.

This problem gives an elementary description of the principle of the "Tethered Satellite System", which has been investigated in Space Shuttle missions as a possible generator of electric power for orbiting systems.