Some properties of Thomson's atom

In Thomson's model for the Hydrogen atom, the positive charge e is uniformly distributed within a sfere of radius a_0. The electron, having charge $-e$, is considered to be a point particle and is located inside the sphere.
a) Find the electric field and the potential generated by the positive charge and the equilibrium position for the electron (assumed to have zero angular momentum).
b) Find the ionization energy U_I (i.e. the energy needed to remove the electron from the atom) and the related value of a_0 consistent with the experimental value $U_I = 13.6$ eV $= 2.18 \times 10^{-18}$ J.
c) Find the oscillation frequency of the electron around its equilibrium position.
d) Find the polarizzability α of the atom and the dielectric constant ϵ in the "solid state" (in which all atoms are adiacent to each other forming a lattice), using the mean field approximation.

Solution

a) The field is radial and can be found by Gauss's theorem applied to an uniformly charged sphere of density $\rho_+ = e/(4\pi a_0^3/3)$:

$$E(r) = \frac{e}{4\pi\epsilon_0} \times \begin{cases} r/a_0^3 & (r < a_0) \\ 1/r^2 & (r > a_0) \end{cases} .$$

The electric potential $V = V(r)$ satisfies $E = -\partial_r V$ and thus we obtain by integrating

$$V(r) = \frac{e}{4\pi\epsilon_0} \times \begin{cases} (3/2a_0 - r^2/2a_0^3) & (r < a_0) \\ 1/r & (r > a_0) \end{cases} ,$$

where we assumed $V(r = \infty) = 0$. The position of stable equilibrium is at $r = 0$.
b) Assuming that the electron is at $r = 0$, its total energy is $-eV(0)$. The energy needed to take the electron to infinity is

$$U_I = eV(0) = \frac{3k_0e^2}{2a_0} ,$$

where $k_0 = 1/4\pi\epsilon_0 \simeq 9 \times 10^9$ SI units. We thus obtain

$$a_0 = \frac{3k_0e^2}{2U_I} = \frac{3 \times 9 \times 10^9 \times (1.6 \times 10^{-19})^2}{2 \times 10^{-18}} \text{ m} = 1.6 \times 10^{-10} \text{ m} .$$

The actual atomic dimensions are about one third of this value.
c) The equation of motion of the electron is

$$\frac{dv_r}{dt} = -\frac{e}{m_e}E(r) = -\frac{k_0e^2}{m_ea_0^3}r \equiv -\omega^2 r ,$$

thus the electron motion is harmonic, with a frequency

$$\omega = \sqrt{\frac{k_0e^2}{m_ea_0^3}} \simeq 7.9 \times 10^{-15} \text{ s} .$$

This value approaches the experimental one just as an order of magnitude estimate.
d) Let us find the electric dipole $\mathbf{p}$ induced by an external field $\mathbf{E}_0$. Due to this latter, the equilibrium position for the electron shifts to the value r_{eq} given by $\mathbf{E}(r_{eq}) = \mathbf{E}_0$, from which we obtain $\mathbf{r}_{eq} = 4\pi\epsilon_0 a_0^3\mathbf{E}_0/e$. The induced dipole is thus

$$\mathbf{p} = -e\mathbf{r}_{eq} \equiv \alpha\mathbf{E}_0 , \qquad \alpha = 4\pi\epsilon_0 a_0^3 .$$

The number density of atoms in the "solid" state is $n = 1/(2a_0)^3$; thus

$$\epsilon_r = 1 + \frac{n\alpha}{\epsilon_0} = 1 + \frac{\pi}{2} .$$