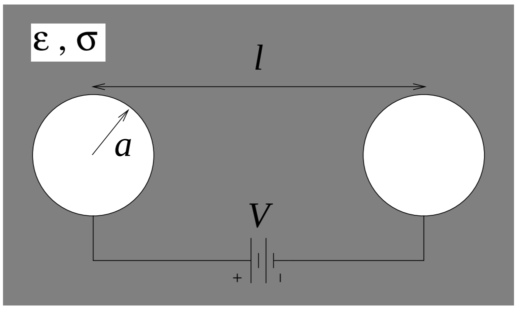
Electrical resistance between two spheres

Two identical, perfectly conducting spheres of radius a are immersed into a fluid of conductivity σ and dielectric constant ϵ. The distance between the centers of the spheres is $\ell \gg a$. A constant voltage difference V is maintained between the spheres by a suitable generator.



As a first approximation assume the charge to be distributed uniformly on each sphere, neglecting electrostatic induction effects.

a) the charge on each sphere,
b) the current I flowing between the spheres and the resistance $R = V/I$.
c) If the voltage generator is disconnected, find the temporal law and the decay time of the discharging of the spheres.
d) Now consider how induction electrostatic effect modify the previous answers, to the lowest order in a/ℓ.

Solution

a) If the charges are distributed uniformly on the surface of the spheres, each sphere generates a field identical to that of a point charge in the center of the sphere. The field generated by each sphere on the other is neglected consistently. For symmetry reasons, the charges will be $\pm Q$ and the potentials $\pm V/2$, respectively. Thus

$$\frac{V}{2} = \frac{Q}{4\pi\epsilon a}, \qquad Q = 2\pi\epsilon a V. \tag{1}$$

b) The current I can be computed as the flux of the current density over any close surface containing one of the spheres. Due to Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ the flux of $\mathbf{J}$ is σ times the flux of $\mathbf{E}$, i.e. σ/ϵ times the charge contained into the surface:

$$I = \int_S \mathbf{J}\cdot d\mathbf{S} = \sigma\int_S \mathbf{E}\cdot d\mathbf{S} = \sigma\frac{Q}{\epsilon} = 2\pi\sigma a V. \tag{2}$$

To check the result, we can also compute I by evaluating the flux of $\mathbf{J}$ on the midplane

$$\int_S \mathbf{J}\cdot d\mathbf{S} = \int_0^\infty \sigma E_x(\rho) 2\pi\rho d\rho, \qquad E_x(\rho) = \frac{2Q}{4\pi\epsilon}\frac{\ell/2}{[(\ell/2)^2+\rho^2]^{3/2}}, \tag{3}$$

$$I = \frac{Q\sigma}{\epsilon}\int_0^\infty \frac{\rho\ell/2}{[(\ell/2)^2+\rho^2]^{3/2}}d\rho = \frac{\sigma}{\epsilon}Q. \tag{4}$$

Thus the resistance does not depend on ℓ (to the lowest order in a/ℓ):

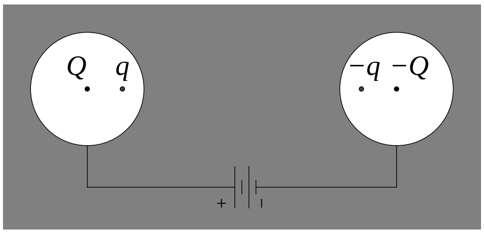
$$R = \frac{V}{I} = \frac{1}{2\pi\sigma a}. \tag{5}$$

c) By applying the law for the charge conservation

$$\frac{dQ}{dt} = -I = -\frac{\sigma}{\epsilon}Q, \qquad Q(t) = Q(0)e^{-t/\tau}, \qquad \tau = \frac{\epsilon}{\sigma}. \tag{6}$$

d)

To the lowest order in a/ℓ electrostatic induction effects can be taken into account by describing the field as produced by two charges $\pm Q$ in the centers of the spheres and two charges $\pm q = \pm(a/\ell)Q$ at a distance $b = a^2/d$ from the centers along the connecting line.



Within this approximation, the potential of each sphere is $\simeq \pm Q/4\pi\epsilon a$, thus $Q \simeq 2\pi\epsilon a V$, and the total induced charges are $\pm(Q+q) = \pm Q(1+a/\ell)$. The flux of $\mathbf{E}$ over any surface containing one of the spheres is now $\pm Q(1+a/\ell)/\epsilon$, respectively, and the resistance becomes $R \simeq [2\pi\sigma a(1+a/\ell)]^{-1}$.

The discharge law and time do not change, since we may write

$$\frac{d(Q+q)}{dt} = -I = -\frac{\sigma}{\epsilon}(Q+q). \tag{7}$$