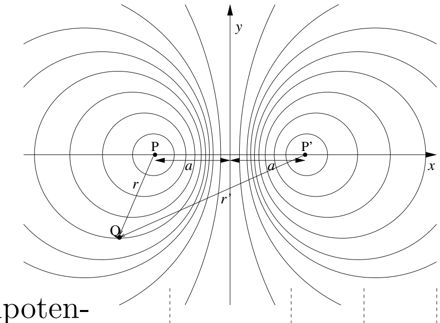
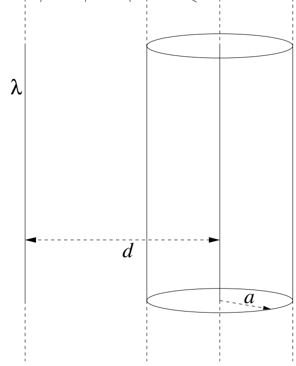
A charged wire in front of a cylindrical conductor



Consider two points P and P' on a plane. For a generic point Q on the plane, let $r = \overline{PQ}$ and $r' = \overline{PQ}'$ the distances from P and P', respectively.

a) Show that the family of curves defined by the equation $r/r' = K$, where K is constant, is a set of circumferences as in the Figure.



b) Use the above obtained geometrical result to show that the equipotential surfaces of the electrostatic field generated from two parallel, infinite wires, having linear charge density λ and $-\lambda$, are cylinders whose section in the plane perpendicular to the wires gives the curves shown at point **a)**.

c) Use the above results to solve the problem of the electrostatic potential generated by an infinite wire having linear charge density λ located in front of an infinite conducting cylinder of radius a. The wire is parallel to the axis of the cylinder and the distance is d, where $d > a$.

Solution

a) Let us take a cartesian frame such that PP' lies on the x axis, $P \equiv (-a, 0)$ and $P' \equiv (a, 0)$. Thus, $r = \sqrt{(x+a)^2 + y^2}$ and $r' = \sqrt{(x-a)^2 + y^2}$, where x and y are the coordinates of Q. Taking the square power of the equation $r/r' = K$ we get

$$\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} = K^2,$$

which can be rearranged as

$$x^2 + y^2 + 2\frac{1+K^2}{1-K^2}ax = -a^2 .$$

On the other hand, the equation of a circumference having its centre in (x_0, y_0) and radius R is

$$(x - x_0)^2 + (y - y_0)^2 = R^2 ,$$

which can be rewritten as

$$x^2 + y^2 - 2x_0 x - 2y_0 y = R^2 - x_0^2 - y_0^2 .$$

From the comparison we see that the curves defined by the equation $r/r' = K$ are circumferences whose centres have coordinates

$$x_0 = \frac{K^2+1}{K^2-1}a, \quad y_0 = 0 ,$$

and whose radii are given by

$$R = \frac{2Ka}{|K^2 - 1|} .$$

We may restrict ourselves to the half-plane $x > 0$, thus $r > r'$, $K > 1$ and we may omit the absolute value sign in the expression for R.

b) From Gauss theorem we know that the electrostatic field and potential generated by an infinite straight wire with linear charge density λ are given by

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}, \qquad V(r) = \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{r}{r_0}\right),$$

where r is the distance from the radius and r_0 an arbitrary constant, corresponding to the distance at which we pose $V = 0$. The potential generated by two wires having charge densities λ and $-\lambda$, respectively, is given by

$$V = \frac{\lambda}{2\pi\varepsilon_0}\log\frac{r}{r_0} - \frac{\lambda}{2\pi\varepsilon_0}\log\frac{r'}{r_0'} = \frac{\lambda}{2\pi\varepsilon_0}\log\frac{r}{r'} + \frac{\lambda}{2\pi\varepsilon_0}\log\frac{r_0'}{r_0},$$

where r_0' is a second arbitrary constant, analogous to r_0. The term

$$\frac{\lambda}{2\pi\varepsilon_0}\log\frac{r_0'}{r_0}$$

can be taken as a single arbitrary constant, which can be set to zero. With this choice the zero of the potential is fixed on the plane $x = 0$. The equation defining the equipotential surfaces is thus

$$\frac{\lambda}{2\pi\varepsilon_0} \log \frac{r}{r'} = V\,,$$

whi leads to

$$\frac{r}{r'} = e^{2\pi\varepsilon_0 V/\lambda}\,.$$

Using this equation and the last formulas of point **a)**, we obtain that the equipotential surfaces are cylinders whose centers are at the points

$$x_0 = \frac{e^{4\pi\varepsilon_0 V/\lambda} + 1}{e^{4\pi\varepsilon_0 V/\lambda} - 1} a, \quad y_0 = 0,$$

and the radii are

$$R = \frac{2e^{2\pi\varepsilon_0 V/\lambda}}{e^{4\pi\varepsilon_0 V/\lambda} - 1} a\,.$$

By multiplying the expressions for x_0 and R for $e^{-2\pi\varepsilon_0 V/\lambda}$ we obtain

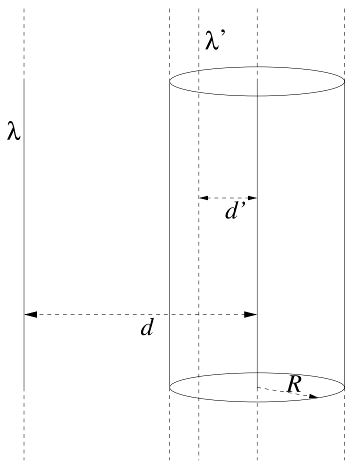
$$x_0 = \frac{e^{2\pi\varepsilon_0 V/\lambda} + e^{-2\pi\varepsilon_0 V/\lambda}}{e^{2\pi\varepsilon_0 V/\lambda} - e^{-2\pi\varepsilon_0 V/\lambda}} a\, R = \frac{2}{e^{2\pi\varepsilon_0 V/\lambda} - e^{-2\pi\varepsilon_0 V/\lambda}} a\,,$$

which can be rewritten in terms of hyperbolic functions

$$x_0 = a \coth\left(\frac{2\pi\varepsilon_0 V}{\lambda}\right), \quad R = \frac{a}{\sinh\left(\frac{2\pi\varepsilon_0 V}{\lambda}\right)}\,.$$

c) We can solve this problem by posing an image wire with charge density $\lambda' = -\lambda$ inside the cylinder, at a position such that, referring to the potential surfaces defined by $\ln(r/r') = \text{cost.}$, the real and the image wires will be located at the points P and P', and the surface of the cylinder will coincide with an equipotential surfaces (this is always possible as far as $d > R$). With this charge configuration the cylinder will be at the costant potential V_0. The explicit solution can be found from the formulae of point **b)**: posing $V' = 2\pi\varepsilon_0 V_0/\lambda$ we obtain the equations

$$2a + d' = d, \qquad a + d' = x_0 = a \coth V', \qquad a/\sinh V' = R$$



which we must solve to find the values of a, d' e V', as a function of d and R. From the first equation we obtain $a = (d - d')/2$, which we substitute in the other two equations to obtain

$$\frac{d + d'}{2} = \frac{d - d'}{2} \coth V', \qquad \frac{d - d'}{2} = R \sinh V'\,.$$

From the latter equation we get $\sinh V' = (d - d')/2R$. Using the identity $\cosh^2 x - \sinh^2 x = 1$ and the definition $\coth x = \cosh x/\sinh x$ we obtain

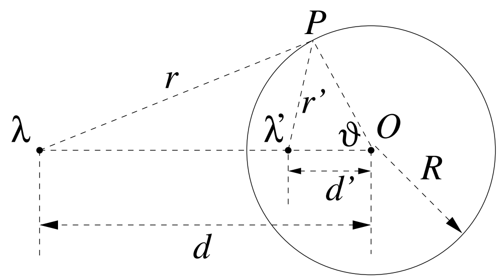
$$\coth V' = \frac{\sqrt{4R^2 + (d-d')^2}}{d - d'},$$

which, after substitution in the first equation, yields

$$\frac{d + d'}{2} = \frac{d - d'}{2} \frac{\sqrt{4R^2 + (d-d')^2}}{d - d'}.$$

Apart from the trivial solution $d' = d$ (the two wires overlap and thus the charge density and the field are zero everywhere) we obtain the solution

$$d' = \frac{R^2}{d}, \qquad a = \frac{d^2 + R^2}{2d}, \qquad V' = \operatorname{arccoth}\left(\frac{d^2 + 3R^2}{d^2 + R^2}\right).$$



Alternatively, may proceed as in the well-know problem of a point charge near a conducting sphere. Let us verify directly that an image wire of linear charge density λ' placed at a proper distance d' from the axis of the cylinder gives the solution to the problem of finding the potential. We must pose the condition that the potential is constant over the whole surface of the cylinder:

$$\frac{\lambda}{2\pi\varepsilon_0} \log r + \frac{\lambda'}{2\pi\varepsilon_0} \log r' = \text{cost.},$$

that gives

$$\lambda \log r + \lambda' \log r' = \text{cost.}$$

Writing r and r' as a function of d, R and ϑ we find

$$\lambda \log\left(\sqrt{d^2 + R^2 - 2Rd\cos\vartheta}\right) + \lambda' \log\left(\sqrt{d'^2 + R^2 - 2Rd'\cos\vartheta}\right) = \text{cost.}$$

Differentiation with respect to ϑ yields

$$\frac{\lambda R d \sin\vartheta}{d^2 + R^2 - 2Rd\cos\vartheta} = -\frac{\lambda R d' \sin\vartheta}{d'^2 + R^2 - 2Rd'\cos\vartheta}$$

We thus find that the sign of λ' must be opposite to the sign of lambda λ. Dividing by R and with some algebra we obtain

$$\lambda d\left(d'^2 + R^2 - 2Rd'\cos\vartheta\right) = -\lambda' d'\left(d^2 + R^2 - 2Rd\cos\vartheta\right)$$

that we may rewrite as

$$\lambda\left(dd'^2 + dR^2 - 2Rdd'\cos\vartheta\right) = -\lambda'\left(d'd^2 + d'R^2 - 2Rdd'\cos\vartheta\right),$$

which is satisfied, indipendently from the value of ϑ, if $\lambda' = -\lambda$ and if

$$dd'^2 + dR^2 - 2Rdd'\cos\vartheta = d'd^2 + d'R^2 - 2Rdd'\cos\vartheta.$$

The term containing $\cos\vartheta$ is cancelled out, and for d' we find, apart from the trivial case $d' = d$, the solution $d' = R^2/d$.