Basics of Laser-Plasma Interaction (lecture 1/3)

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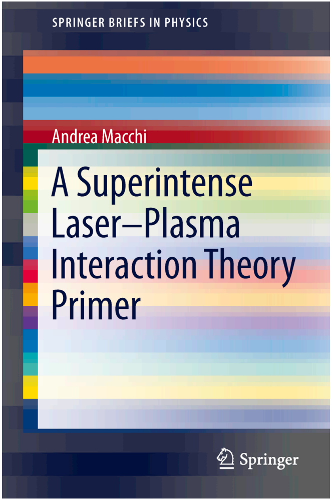
A Compact Reference

A. Macchi,
A Superintense Laser-Plasma Interaction Theory Primer
(Springer, 2013)



Outline of Lecture 1

Warm-up: single electron dynamics

- Relativistic motion in a plane wave
- Ponderomotive force

From one to many electrons: basic equations

- the Vlasov-Maxwell system
- the “cold” fluid equations

Wake waves

- electrostatic waves
- laser wakefield
- wavebreaking

Outline of Lecture 2

Nonlinear “*relativistic*” optics

- Review of linear EM waves in a plasma
- Self-induced transparency
- Self-focusing

Moving mirrors

- Basic formulas
- High harmonics
- “Flying mirrors” from plasma wakes
- Light sails

Outline of Lecture 3

Light sail acceleration

- heating vs radiation pressure
- effects of laser polarization
- 3D effects

Radiation friction

- foundations of the problem
- classical Landau-Lifshitz theory
- macroscopic effects: magnetic field generation

Single electron in a plane wave

An EM plane wave can be described by the vector potential:

$$\mathbf{A}(x,t) = \mathbf{A}(x-ct) \quad \longrightarrow \quad \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{A} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}$$

(we assume propagation along the x-axis)

Equations of Motion (EoM):

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v} = \frac{\mathbf{p}}{m_e\gamma} \,, \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -e\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right]$$

$$\mathbf{r} = \mathbf{r}(t) \qquad \mathbf{p} = \mathbf{p}(t) \qquad \gamma = (\mathbf{p}^2 + m_e^2 c^2)^{1/2} = (1 - \mathbf{v}^2/c^2)^{-1/2}$$

The EoM are nonlinear because of the $\mathbf{v} \times \mathbf{B}$ term and the dependence of the fields on the instantaneous position:

$$\mathbf{E} = \mathbf{E}(\mathbf{r}(t), t) \qquad \mathbf{B} = \mathbf{B}(\mathbf{r}(t), t)$$

When is the motion relativistic?

(Quasi-)Monochromatic wave $\mathbf{A}(x,t) = \mathrm{Re}\left[\hat{\mathbf{A}}(x,t)\mathrm{e}^{ikx-i\omega t}\right]$

(with $\hat{A}(x,t)$ a slowly varying envelope, i.e. the wavepacket profile)

$$\text{Assume} \quad |\mathbf{v}| \ll c \quad \Rightarrow \quad |\mathbf{r}| \ll \lambda = \frac{2\pi c}{\omega} \quad \Rightarrow \quad k|\mathbf{r}| = 2\pi\frac{|\mathbf{r}|}{\lambda} \simeq 0$$

$$\Rightarrow \mathbf{E}(\mathbf{r}(t),t) = \mathbf{E}(kx(t),t) \simeq \mathbf{E}(x=0,t) \quad \text{and} \quad \frac{\mathbf{v}}{c}\times\mathbf{B} \simeq 0$$

$$\text{Solution} \quad \mathbf{p}(t) \simeq \frac{e}{c}\mathbf{A}(0,t) \propto \mathrm{e}^{-i\omega t} \qquad \frac{|\mathbf{v}|}{c} = \frac{p}{m_e c} = \frac{eA_0}{m_e c^2} \equiv a_0$$

The motion becomes relativistic and nonlinear when $a_0 \gtrsim 1$

$$a_0 = 0.85\left(\frac{I\lambda^2}{10^{18}\ \mathrm{W\,cm^{-2}}}\right)^{1/2} \quad \text{where} \quad I \equiv \langle|\mathbf{S}|\rangle = \left\langle\frac{c}{4\pi}|\mathbf{E}\times\mathbf{B}|\right\rangle$$

Constants of motion in a plane wave

Symmetry properties of the EoM $\longrightarrow$ conserved quantities:

$$\mathbf{p}_\perp - \frac{e}{c}\mathbf{A} = \mathbf{C}_1 \qquad p_x - m_e\gamma c = C_2$$

("$\perp$" denotes the transverse direction, i.e. yz plane)
Assuming as initial conditions $\mathbf{p} = 0$, $\mathbf{A} = 0$ i.e. $\mathbf{C}_1 = 0$, $C_2 = -m_e c$

$$p_x = \frac{\mathbf{p}_\perp^2}{2m_e c} = \frac{1}{2m_e c}\left(\frac{e}{c}\mathbf{A}\right)^2$$

After the EM pulse is gone $\mathbf{A} = 0$ again $\Rightarrow p_x = 0$
$\Rightarrow$ no acceleration by EM plane wave in vacuum
("Lawson-Woodward" theorem)

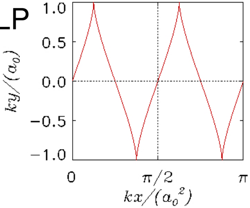
Solutions for a plane monochromatic wave

$$\mathbf{A}(x,t) = A_0\left[\hat{\mathbf{y}}\cos\theta\cos(kx-\omega t) - \hat{\mathbf{z}}\sin\theta\sin(kx-\omega t)\right]$$

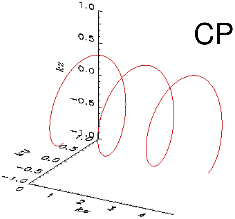
with $\mathbf{C}_1 = 0$, $C_2 = -m_e c$ (*adiabatic* field rising in an infinite time)

$\theta = 0, \pm\frac{\pi}{2}$
linear
polarization
(LP)

LP



CP



$\theta = \pm\frac{\pi}{4}$
circular
polarization
(CP)

Constant longitudinal drift: $\langle p_x\rangle = m_e c a_0^2/4$, $\langle v_x\rangle = c a_0^2/(a_0^2+4)$
(origin: absorption of EM energy $\propto$ absorption of EM momentum)
Notice: $\gamma = \gamma(t)$ for LP, $\gamma = (1+a_0^2/2)^{1/2} = \text{const.}$ for CP

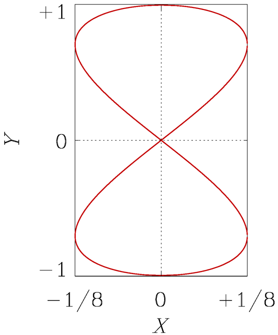
Figure of Eight

For LP, switch to the frame where $\langle v_x \rangle = 0$
i.e. take $\mathbf{C}_1 = 0$, $C_2 = m_e \gamma_0 c$

Closed trajectory $16X^2 = Y^2(1 - Y^2)$

$$X \equiv \frac{\gamma_0}{a_0^2} kx \qquad Y \equiv \frac{\gamma_0}{a_0} ky$$

The trajectories are *self-similar*



Messages for the study of electrons in superintense fields:

- initial conditions are crucial
- polarization matters
- EM field properties constrain the dynamics

Ponderomotive approximation

Aim: describe the motion in a *quasi-periodic* field $(T = 2\pi/\omega)$

$$\mathbf{A}(\mathbf{r}, t) = \mathrm{Re}\left[\tilde{\mathbf{A}}(\mathbf{r}, t)\mathrm{e}^{-i\omega t}\right]$$

for which the average over a period yields $\left(\langle f\rangle \equiv T^{-1}\int_0^T f(t')\mathrm{d}t'\right)$

$$\langle\mathbf{A}(\mathbf{r}, t)\rangle \simeq 0 \qquad \left\langle\tilde{\mathbf{A}}(\mathbf{r}, t)\right\rangle \simeq \tilde{\mathbf{A}}(\mathbf{r}, t)$$

Idea: find an EoM for the “slow” (period-averaged) motion

$$\mathbf{r}(t) \equiv \mathbf{r}_s(t) + \mathbf{r}_o(t) \qquad \langle\mathbf{r}_o(t)\rangle \simeq 0 \qquad \langle\mathbf{r}_s(t)\rangle \simeq \mathbf{r}_s(t)$$

analogous to the *guiding center approximation* in a non-uniform magnetic field

Ponderomotive force

A *perturbative*, *non-relativistic* approach including lowest order contributions from the $\mathbf{v}\times\mathbf{B}$ term and the spatial variation of $\mathbf{E}=\mathbf{E}(\mathbf{r},t)$ yields the EoM for $\mathbf{v}_s(t)=\langle\mathbf{v}(t)\rangle$ and $\mathbf{r}_s(t)=\langle\mathbf{r}(t)\rangle$

$$m_e\frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t}=-\frac{e^2}{2m_e\omega^2}\nabla\left\langle\mathbf{E}^2(\mathbf{r}_s(t),t\right\rangle\equiv\mathbf{f}_p \qquad \frac{\mathrm{d}\mathbf{r}_s}{\mathrm{d}t}=\mathbf{v}_s$$

Relativistic extension (slightly controversial):

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_{\mathrm{eff}}\mathbf{v}_s)=-\nabla(m_{\mathrm{eff}}c^2)\equiv\mathbf{f}_p$$

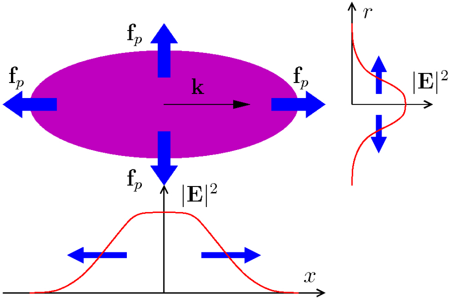
$$m_{\mathrm{eff}}\equiv m_e(1+\left\langle\mathbf{a}^2\right\rangle)^{1/2} \qquad \left(\mathbf{a}\equiv e\mathbf{A}/m_ec^2\right)$$

The (time- and space-dependent) effective mass m_{eff} accounts for relativistic inertia due to the oscillatory motion

Ponderomotive effects

$$\mathbf{f}_p \propto -\nabla|\mathbf{E}|^2$$

$\Rightarrow$ electrons are scattered off higher field regions
A laser pulse (of finite length and width) accelerates electrons both longitudinally and radially



Notice: we *define* $\mathbf{f}_p$ as a *secular*, “slow” force (it does *not* include oscillating nonlinear terms)
The ponderomotive force concept is tightly related to that of radiation pressure

Many particles: kinetic equation

Distribution function $f_a = f_a(\mathbf{r}, \mathbf{p}, t)$ (species index $a = e, i, \ldots$): probability density in phase space cell $\mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{p}$

$$\frac{\mathrm{d}f_a}{\mathrm{d}t} = \frac{\partial f_a}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\dot{\mathbf{r}}_a f_a) + \frac{\partial}{\partial \mathbf{p}}(\dot{\mathbf{p}}_a f_a) = 0 \tag{1}$$

$$\dot{\mathbf{r}}_a = \mathbf{v} = \frac{\mathbf{p}c}{(\mathbf{p}^2 + m_a^2 c^2)^{1/2}} \qquad \dot{\mathbf{p}}_a = q_a\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \qquad \begin{array}{l} \mathbf{E} = \mathbf{E}(\mathbf{r}, t) \\ \mathbf{B} = \mathbf{B}(\mathbf{r}, t) \end{array}$$

Eq.(1) is a continuity equation in phase space, valid when

- particle number is conserved (no ionization, no particle production, ...)
- *collisions* are negligible (if $\mathbf{E}$ and $\mathbf{B}$ are mean fields)

Maxwell-Vlasov system

Coupling of Eq.(1) to Maxwell's equations:

$$\rho(\mathbf{r},t) = \sum_a q_a \int f_a \mathrm{d}^3 p \qquad \mathbf{J}(\mathbf{r},t) = \sum_a q_a \int \mathbf{v} f_a \mathrm{d}^3 p$$

$$\nabla\cdot\mathbf{E} = 4\pi\rho \qquad \nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{B} \qquad \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial_t\mathbf{E} \quad (2)$$

The system of Eqs.(1)+(2) is *not* the most fundamental but the "*best*" suitable approach to a classical laser-plasma environment (with possible extensions to include ionization, radiation friction, particle production, ...)
Most of the times Eqs.(1)+(2) are tackled numerically, typically with Particle-In-Cell (PIC) codes

Cold fluid equations

Phylosophy of fluid approach: describing the plasma in terms of quantities *averaged* over the momentum distribution

$$n_a = n_a(\mathbf{r}, t) \equiv \int f_a(\mathbf{r}, \mathbf{p}, t)\mathrm{d}^3 p \qquad \mathbf{p}_a = \mathbf{p}_a(\mathbf{r}, t) \equiv n_a^{-1} \int \mathbf{p} f_a(\mathbf{r}, \mathbf{p}, t)\mathrm{d}^3 p$$

A finite set of equations can be only obtained by “truncation”, i.e. an assumption of $f_a(\mathbf{r}, \mathbf{p}, t)$. If the coherently driven motion is more important than the random thermal motion then

$$\partial_t n_a + \nabla \cdot (n_a \mathbf{u}_a) = 0 \qquad (\partial_t + \mathbf{u}_a \cdot \nabla)\mathbf{p}_a = q_a(\mathbf{E} + \mathbf{u}_a \times \mathbf{B})$$

$$\left(\mathbf{u}_a = \frac{\mathbf{p}_a c}{(\mathbf{p}_a^2 + m_a^2 c^2)^{1/2}}\right).$$

Pressure is neglected (“cold” plasma)

Linear waves in a plasma

General wave equation for $\mathbf{E}$ from Maxwell's equations:

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\mathbf{E} - \nabla(\nabla\cdot\mathbf{E}) = \frac{4\pi}{c^2}\partial_t\mathbf{J}$$

Assume monochromatic fields $\mathbf{E}(\mathbf{r},t) = \tilde{\mathbf{E}}(\mathbf{r})\mathrm{e}^{-i\omega t}$ *et cetera*
Using linearized, non-relativistic equations ($|\mathbf{u}_e| \ll c$)

$$\partial_t\mathbf{u}_e = -\frac{e}{m_e}\mathbf{E} \qquad \mathbf{J} = -en_e\mathbf{u}_e \qquad \text{(ions taken at rest)}$$

$$\tilde{\mathbf{J}} = -i\frac{n_e e^2}{m_e\omega}\tilde{\mathbf{E}} = -\frac{i}{4\pi}\frac{\omega_p^2}{\omega}\tilde{\mathbf{E}}, \qquad \omega_p \equiv \left(\frac{4\pi e^2 n_e}{m_e}\right)^{1/2}$$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{E}} - \nabla(\nabla\cdot\tilde{\mathbf{E}}) = \frac{\omega_p^2}{c^2}\tilde{\mathbf{E}} \qquad \text{Helmoltz equation}$$

Linear transverse (EM) waves

Taking $\tilde{\mathbf{E}}(\mathbf{r}) = E_0 \boldsymbol{\epsilon} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$, $\quad \nabla\cdot\mathbf{E} = 0,\quad \mathbf{k}\cdot\boldsymbol{\epsilon} = 0 \quad \mathbf{B} = \mathbf{k}\times\mathbf{E}/k$

$$\left(\nabla^2 + \varepsilon(\omega)\frac{\omega^2}{c^2}\right)\tilde{\mathbf{E}} = \left(\nabla^2 + \mathrm{n}^2(\omega)\frac{\omega^2}{c^2}\right)\tilde{\mathbf{E}} = 0$$

$\varepsilon(\omega) = \mathrm{n}^2(\omega) = 1 - \dfrac{\omega_p^2}{\omega^2}$ $\quad \varepsilon(\omega)$ dielectric function, $\mathrm{n}(\omega)$ refractive index

dispersion relation $\quad k^2c^2 = \varepsilon(\omega)\omega^2 = \omega^2 - \omega_p^2$

Phase and group velocities (assuming $\omega > \omega_p$ i.e. k real)

$$v_p = \frac{\omega}{k} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} > c \qquad v_g = \frac{\partial\omega}{\partial k} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} < c$$

Linear longitudinal electrostatic (ES) waves

Taking $\tilde{\mathbf{E}}(\mathbf{r}) = E_0 \boldsymbol{\epsilon} e^{i\mathbf{k}\cdot\mathbf{r}}$, $\mathbf{k} \parallel \boldsymbol{\epsilon} = 0$, $\nabla \cdot \mathbf{E} = \mathbf{k} \cdot \mathbf{E}$, $\mathbf{B} = 0$

$$\nabla(\nabla \cdot \tilde{\mathbf{E}}) = \nabla^2 \tilde{\mathbf{E}} \quad \Rightarrow \quad (\omega^2 - \omega_p^2)\tilde{\mathbf{E}} = 0 \quad \Rightarrow \quad \omega = \omega_p$$

Wavevector k and phase velocity $v_p = \omega_p / k$ are not constrained by dispersion relation

Note: we still assume the thermal velocity to be negligible i.e.

$$v_{\rm osc} = \frac{qE_0}{m_e \omega_p} \gg v_{\rm th} = \left(\frac{T_e}{m_e}\right)^{1/2}$$

in the opposite regime ($v_{\rm th} \gg v_{\rm osc}$) $\qquad \omega^2 = \omega_p^2 + 3k^2 v_{\rm th}^2$

Example of acceleration by a wave

From: T.Katsouleas, Nature **444** (2006) 688

Looking for the perfect wave for electrons

- A longitudinal wave is needed
- Phase velocity should be tuned to match those of particles (so $v_p \lesssim c$ for relativistic acceleration)
- Highest amplitude is desirable

Electrostatic plasma waves are good candidates:

- Wave is longitudinal ($\mathbf{E} \parallel \mathbf{k}$)
- No "breakdown" limit as in conventional accelerators
- Phase velocity $v_p = \omega/k = \omega_p/k$ may be tuned "by construction"

Wake waves

A force pulse traveling in a plasma with velocity v_f excites a wake of plasma oscillations with phase velocity $v_p = v_f$

Example: a charge bunch penetrating a plasma loses its energy to the wake (collective stopping)

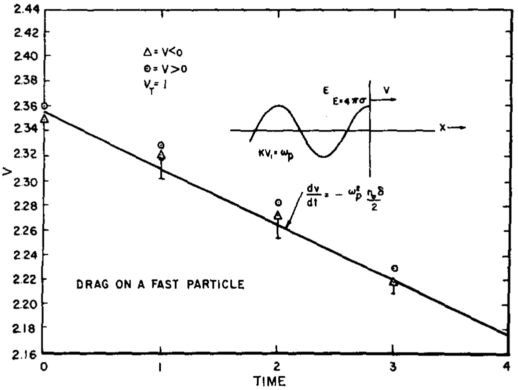


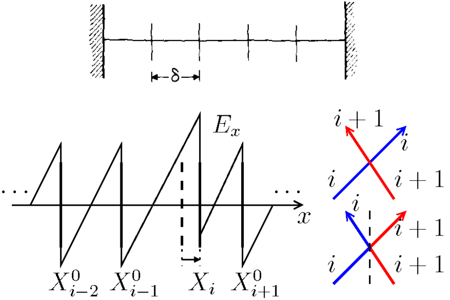
FIG. 6. The drag on a fast sheet.

J. Dawson, Phys. Fluids **5** (1962) 445

Simulating wakes with “Dawson’s sheet” model - I

1D ES model with “macro-electrons” as charge sheets over a neutralizing background

EoM for i-th sheet position $X_i = X_i(t)$ (if the ordering $X_i < X_{i+1}$ is preserved!)



$$\frac{\mathrm{d}^2 X_i}{\mathrm{d}t^2} = -\frac{e}{m_e}E_x(X_i) + \frac{f_{\mathrm{ext}}}{m_e} = -\omega_p^2\left(X_i - X_i(0)\right) + \frac{f_{\mathrm{ext}}}{m_e} \tag{3}$$

Sheet crossing is equivalent to a “reindexing” of sheets $\longrightarrow$ Eq.(3) holds with the swap of indices playing the nonlinear interaction!

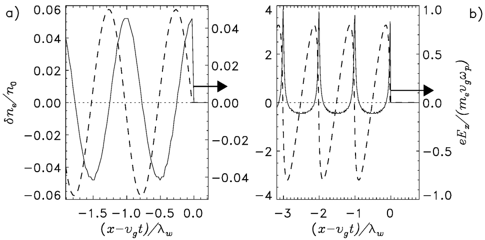
Simulating wakes with "Dawson's sheet" model - II

Linear solution for weak, impulsive force $f_{\text{ext}} = m_e u_0 \delta(t - x/v_f)$

$$u_x \simeq u_0 \Theta(\tau) \cos \omega_p \tau \qquad E_x \simeq \frac{m_e \omega_p u_0}{e} \Theta(\tau) \sin \omega_p \tau$$

$$\delta n_e \simeq n_0 \frac{u_0}{v_f} \Theta(\tau) \cos \omega_p \tau \qquad \tau = t - \frac{x}{v_f}$$

The forcing is strong for $u_0 \to v_f$
i.e. $\delta n_e \to n_0$
n_e becomes spiky and E_x sawtooth



Laser wakefield

Ponderomotive force of short laser pulse
$\longrightarrow$ travelling force at the group velocity of an EM wave:

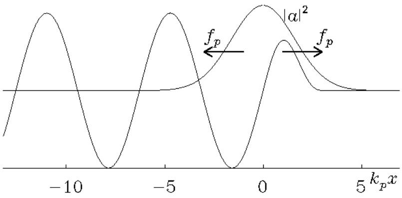
$$f_p(x,t) = f_p(x - v_g t) = -m_e c^2 \nabla (1 + \langle \mathbf{a}^2 (x - v_g t) \rangle)^{1/2}$$

$$v_g = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \lesssim c \quad (\omega_p \ll \omega)$$

Optimal pulse duration

$$\tau_p \simeq \pi/\omega_p$$

$$E_w \simeq (m_e c a_0/e\tau_p) \quad (a_0 \gg 1)$$



Tajima & Dawson, Phys. Rev. Lett **43** (1979) 267

Wavebreaking

The electron density must remain positive:

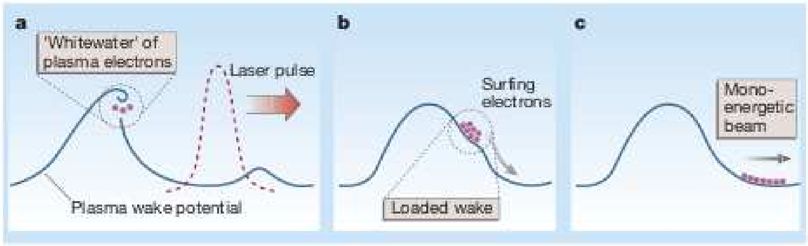
$$n_e = n_0 + \delta n_e > 0 \quad \Leftrightarrow \quad |\delta n_e| < n_0$$

$\delta n_e \longrightarrow n_0$ as $u_0 \longrightarrow v_p$: "self-acceleration" of the wave electrons
$\Rightarrow$ singularity in density profile, "breaking" of the wave
Electric field threshold from nonlinear relativistic theory
[Akhiezer & Polovin, Sov. Phys. JETP **3** (1956) 696]

$$E_{\text{max}} = \frac{m_e c \omega_p}{e} \left(2(\gamma_p - 1)\right)^{1/2} \qquad \gamma_p = \left(1 - \frac{v_p^2}{c^2}\right)^{-1/2}$$

limiting factor in Tajima & Dawson's original proposal
(non-relativistic: $E_{\text{max}} \simeq m_e v_p \omega_p / e$)

Electron acceleration near wavebreaking threshold



[T. Katsouleas, Nature **431** (2004) 515]

Onset of wavebreaking leads to self-injection of electrons

Wakefield is driven to maximum amplitude: creation of a cavity or "bubble" behind the laser pulse