

Fundamentals of Plasma Physics (A Plasma Theory Primer)

Andrea Macchi

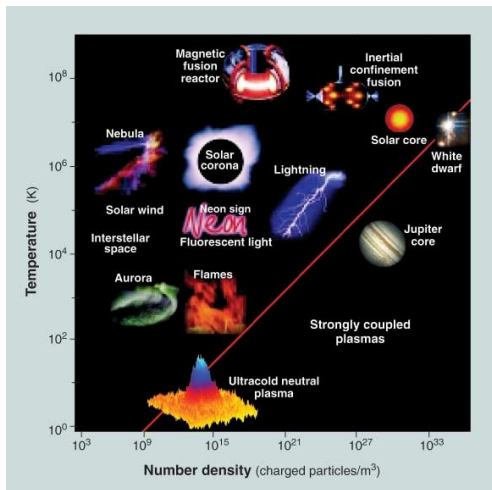
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SP2ARK Seasonal School
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Plasmas across the universe (and on Earth)



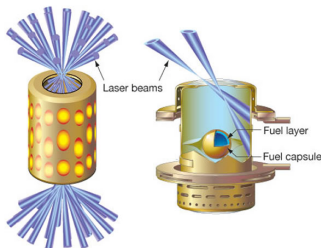
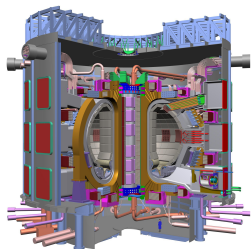
“>99% of the visible universe is plasma”

Plasmas span 30 & 9 orders of magnitude in temperature and density respectively

figure from T. C. Killian, “Ultracold Neutral Plasmas”, *Science* **316** (2007) 705

A bad reputation from fusion?

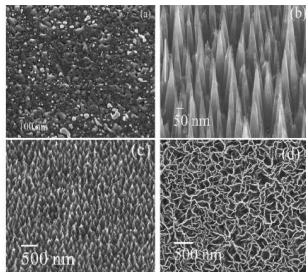
Much of laboratory plasma physics has grown around the quest for controlled thermonuclear fusion (“always 50 years away” BUT some significant progress in the last four years . . .)



“Is plasma involved? It can't work”
(Edward Teller on an early proposal of controlled fusion)



Plasmas that work: discharges

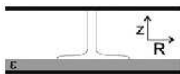
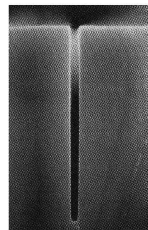
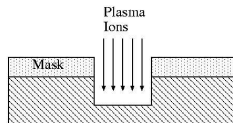


Plasma etching
e.g. for microprocessor
fabrication

M.A.Lieberman & J.L.Lichtenberg
*Principles of Plasma Discharges
& Materials Processing* (2005)

Plasma nanofabrication

K.Ostrikov *Rev.Mod.Phys.* **77** (2005) 489



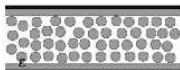
(a)



(b)



(c)



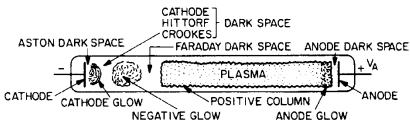
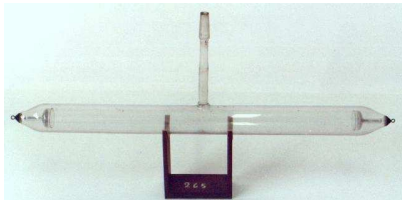
(c)

Dielectric Barrier Discharges
for plasma displays, actuators,
surface processing, ozone
production, . . .

(d)

V.I.Gibalov & G.J.Pietsch *J.Phys.D:Appl.Phys.* **33** (2000) 2618

In the beginning there was the discharge tube ...

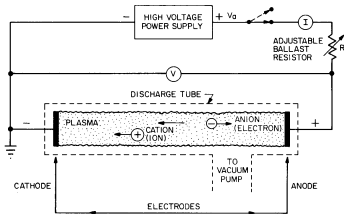


"The positive column [...] is what Irving Langmuir had in mind when he gave the definition of a plasma"
(J. Reece Roth, *Industrial Plasma Engineering*, 2004)

I. Langmuir: Nobel prize,
Chemistry (1932)

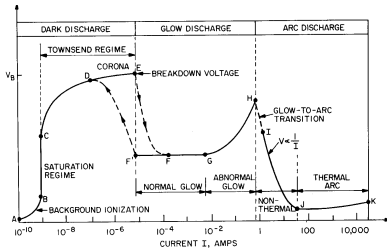


*"There are more things
between cathode and anode than
are dreamt in your philosophy"*
(H. Raether)

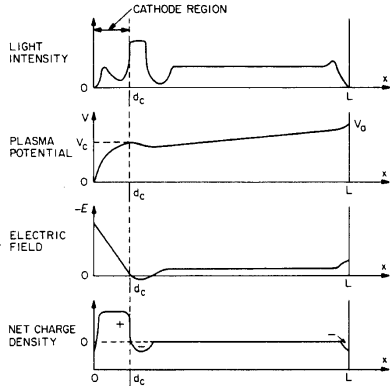
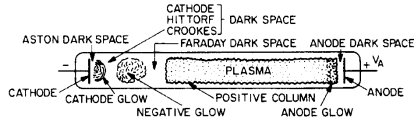


... and the tube is already a complex object

- regions of different luminosity
- nonlinear $V-I$ characteristics (including negative differential resistance, hysteresis, ...)



- effects of **space charge** and **electrostatic field** near the cathode



The current in a vacuum diode

Electrons emitted from the hot cathode are accelerated by the electric field between the electrodes $\mathbf{E} = E(x)\hat{\mathbf{x}}$ generated by the external generator and the charge density $\rho(x) = -en_e(x)$

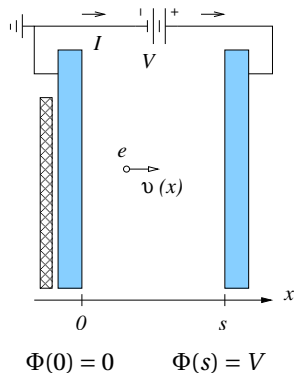
$\Phi(x)$: electric potential

$\mathbf{E} = -\nabla\Phi$ i.e. in 1D $E(x) = -\partial_x\Phi(x)$

$\partial_x^2\Phi(x) = -4\pi\rho(x)$ (Poisson's equation)

Electron energy conservation (assuming $v(0) \simeq 0$)

$$K + U = \frac{m_e}{2} v^2(x) - e\Phi(x) = 0 \quad v(x) = \left(\frac{2e\Phi(x)}{m_e} \right)$$



The current in a vacuum diode

Constant current density

$$\frac{I}{A} = -J = -en_e(x)v(x)$$

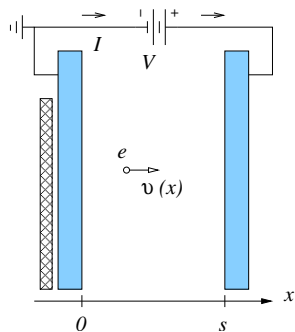
$$\partial_x^2 \Phi(x) = 4\pi en_e(x) = \frac{4\pi I}{Av(x)}$$

$$= -\frac{4\pi}{\sqrt{2}} \frac{I}{A} \left(\frac{m_e}{e\Phi(x)} \right)^{1/2}$$

$$\Phi(x) = \left(\frac{3^4 \pi^2}{2} \right)^{1/3} \left(\frac{m_e}{e} \right)^{1/3} \left(\frac{I}{A} \right)^{2/3} x^{4/3}$$

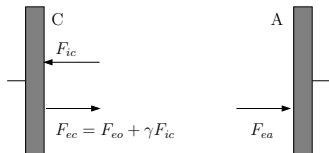
Posing $\Phi(s) = V$ we obtain the **nonlinear** $V - I$ characteristics

$$I = \frac{\sqrt{2}}{9\pi} \left(\frac{e}{m_e} \right)^{1/2} \frac{A}{s^2} V^{3/2} \quad (\text{Child's law})$$



Producing a plasma discharge

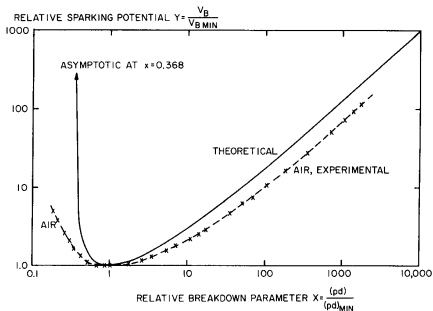
Electrical breakdown depends on distance d and voltage drop V_b between electrodes, gas pressure p and secondary electron emission coefficient from the cathode γ that provides a “positive” feedback:



$$V_b = \frac{\mathcal{B}pd}{\ln[\mathcal{A}Pd / \ln(1 + 1/\gamma)]}$$

(Paschen's law)

Accurate modeling must also take into account the space charge effects near the cathode



Plasma sheath

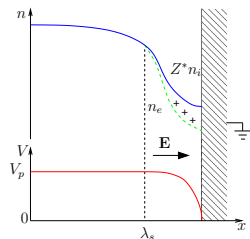
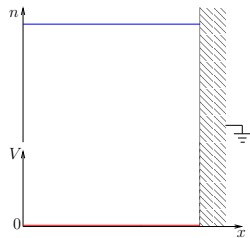
Exposing a grounded conductor to a discharge leads to the formation of a positive charge separation region (sheath) due to larger electron mobility

→ ions are accelerated towards the surface with $v_i \simeq \left(\frac{Z k_B T_e}{m_i} \right)^{1/2}$ (Bohm velocity)

T_e : electron temperature

- sheath field determines ion “landing” energy: highly relevant for surface processing application

- the mathematical modeling is not trivial (large bulk of theoretical and sometimes controversial publications ...)



Temperature in a plasma

For a plasma in thermal equilibrium the ionization degree $\alpha = n_{\text{ion}}/n_0$ can be calculated as a function of $k_B T$ (Quantum & statistical mechanics required with some subtle issues) [see R. Peierls *Surprises in Theoretical Physics* (1979)]

For H with ionization potential $I = 13.6$ eV

$$\frac{\alpha^2}{1 - \alpha} = \frac{2}{(2\pi)^{3/2}} \frac{(k_B T)^{3/2}}{n_0 \hbar^2} e^{-I/k_B T} \quad (\text{Saha equation})$$

$$\alpha \sim 10^{116} @ T = 300 \text{ K}, \quad \alpha \sim 0.5 @ T = 2.2 \times 10^4 \text{ K}$$

BUT most plasmas are far from thermal equilibrium!

- plasmas can be created even at a few K by photoionization
- T_e and T_i are taken as average energies (often $T_e \neq T_i$)

Looking for a general definition of plasma (if any)

- *The fourth state of matter*

(but most of laboratory plasmas are neither created via a smooth phase transition nor are in thermal equilibrium, and plasmas exist also at few Kelvin temperature)

- *A medium with ionized atoms and free electrons*

(if we do not restrict to gas densities also the “Fermi sea” of electrons in a metal is a (quantum degenerate) plasma)

- *An ensemble of particles interacting via long-range interactions and exhibiting collective behavior*

(if we do not restrict to (sub-)atomic particles and EM interactions also a galaxy is a (gravitational) plasma)

Collective behavior

The motion of individual electrons and ions (position $\mathbf{r}_a = \mathbf{r}_a(t)$, velocity $\mathbf{v}_a = \mathbf{v}(t)$, $a = e, i$) is determined by the Lorentz force in the EM fields $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ & $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$:

$$m_a \frac{d\mathbf{v}_a}{dt} = q_a \left(\mathbf{E} + \frac{\mathbf{v}_a}{c} \times \mathbf{B} \right)_{\mathbf{r}=\mathbf{r}_a(t)} \quad \mathbf{v}_a = \frac{d\mathbf{r}_a}{dt}$$

In general \mathbf{E} and \mathbf{B} are the sum of “external” fields (produced by electrodes, coils, radiation, . . .) plus those **self-generated** by the charge ($\rho = \rho(\mathbf{r}, t)$) and current ($\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$) densities in the plasma, to be determined **self-consistently** with the position and velocity of all the particles.

Example: space-charge effects in the vacuum diode or the discharge tube

Plasma oscillations in a metal slab

A small displacement δ of *all* the electrons generates surface charge densities $\pm\sigma = \pm en_e\delta$
Self-generated electric field (capacitor-like)

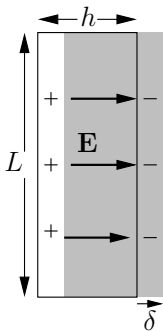
$$\mathbf{E} = 4\pi\sigma\hat{\mathbf{x}} = 4\pi en_e\delta$$

Equation of motion for electrons

$$m_e \frac{d^2\delta}{dt^2} = -e\mathbf{E} = -4\pi en_e\delta$$

Solution: $\delta(t) = \delta(0) \cos\omega_p t$

$$\omega_p = \left(\frac{4\pi n_e e^2}{m_e} \right)^{1/2} \quad \text{plasma frequency}$$



Plasma oscillations in a metal sphere

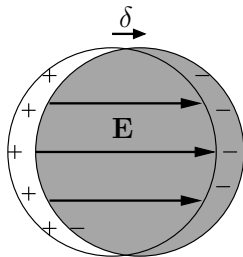
Self-generated electric field

$$\mathbf{E} = \frac{4\pi}{3}\sigma\hat{\mathbf{x}} = 4\pi en_e\delta\hat{\mathbf{x}}$$

(hint: consider the overlap of two identical spheres of opposite charge density and use the superposition principle)

Equation of motion for electrons

$$m_e \frac{d^2\delta}{dt^2} = -e\mathbf{E} = -\frac{4\pi}{3}en_e\delta$$



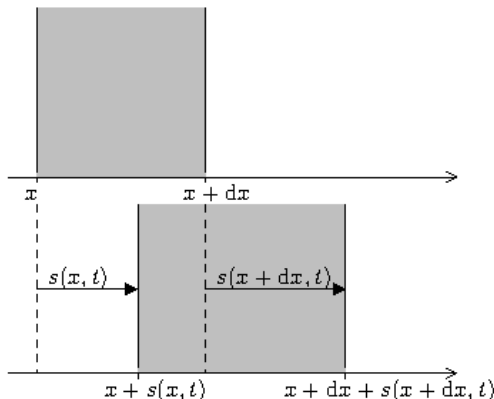
Solution: $\delta(t) = \delta(0) \cos \omega_M t$ with $\omega_M \equiv \frac{\omega_p}{\sqrt{3}}$ Mie frequency

Plasma oscillations in a “cold” homogeneous plasma

1D displacement of electrons $s(x, t)$
velocity $v(x, t) = \partial_t s(x, t)$
initial density n_0

Assumptions:

- $v(x, t) \gg$ thermal velocity
- immobile ions
- electrons starting from different positions do not overtake each other



(see e.g. *The Feynman Lectures on Physics*, vol.2, sec.7-3)

Plasma oscillations in a “cold” homogeneous plasma

$$n_0 dx = n_e(x, t) [x + dx + s(x + dx, t) - x - s(x, t)]$$

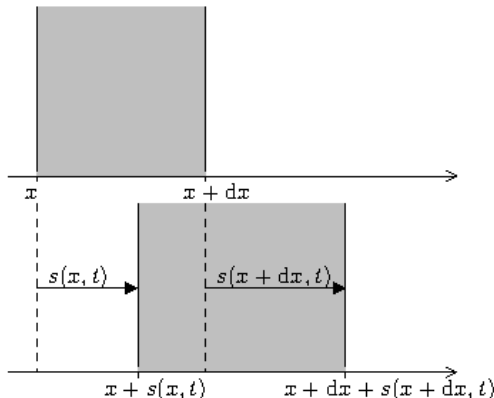
(charge conservation)

$$n_e(x, t) = \frac{n_0}{1 + \partial_x s(x, t)}$$
$$\simeq n_0 [1 - \partial_x s(x, t)]$$

assuming $|\partial_x s(x, t)| \ll 1$
i.e. small displacements

Electric field
from Gauss's law

$$\partial_x E_x = 4\pi e(n_e - n_0)$$
$$E_x = E_x(x, t) = 4\pi n_0 e s(x, t)$$



Plasma oscillations in a “cold” homogeneous plasma

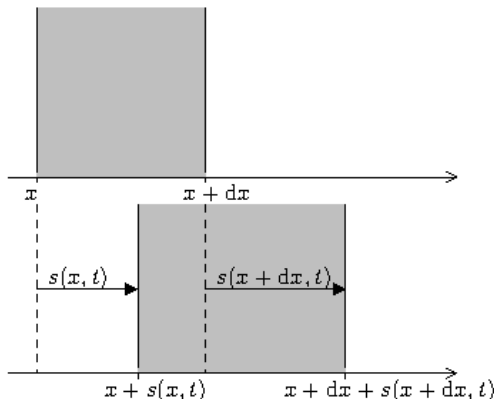
Equation of motion

$$\begin{aligned}\partial_t^2 s(x, t) &= -\frac{e}{m_e} E_x(x, t) \\ &= -\frac{4\pi n_0 e^2}{m_e} s(x, t)\end{aligned}$$

General solution

$$s(x, t) = \text{Re} \left[\tilde{s}(x) e^{-i\omega_p t} \right]$$

(localized oscillation)



Boltzmann relation

Assume thermal equilibrium and that for species a the electric forces are balanced by pressure gradients ($q_a n_a \mathbf{E} - \nabla P_a = 0$) and $P_a = n_a k_B T_a$ (perfect gas)

$$\mathbf{E} = -\nabla\Phi \quad \longrightarrow \quad q_a \Phi = -k_B T_a \ln\left(\frac{n_a}{n_0}\right)$$

$$\longrightarrow \quad n_a = n_0 \exp\left(\frac{q_a \Phi}{k_B T_a}\right) \quad (\text{Boltzmann equilibrium condition})$$

Self-consistent Poisson equation for $\Phi = \Phi(\mathbf{r})$

$$\nabla^2 \Phi = -4\pi\rho = -4\pi \sum_a q_a n_0 \exp\left(\frac{q_a \Phi}{k_B T_a}\right)$$

Electric field screening in a capacitor

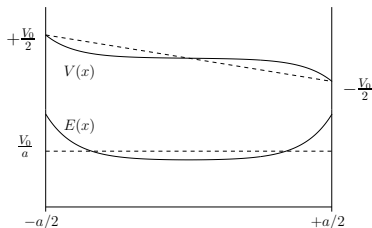
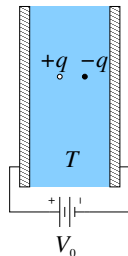
Globally neutral plasma of opposite charge species ($\pm q$) inside a plane capacitor with voltage drop V_0
Assume $q\Phi \ll k_B T \rightarrow$ linearize Poisson equation

$$\partial_x^2 \Phi = -4\pi q n_0 \left(e^{-q\Phi/k_B T} - e^{+q\Phi/k_B T} \right) \simeq \frac{8\pi q^2 n_0}{k_B T} \Phi$$

$$\Phi(x) = -\frac{V_0 \sinh(x/\sqrt{2})\lambda_D}{2 \sinh(a/\sqrt{2}\lambda_D)}$$

$$\lambda_D \equiv \left(\frac{k_B T}{4\pi e^2 n_0} \right)^{1/2}$$

The field is screened over a distance λ_D (Debye length)

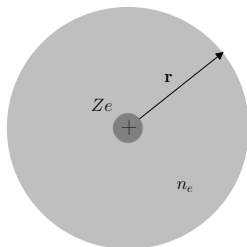


Electric field screening between plasma particles

Apply linearized Poisson-Boltzmann equation around a point-like charge $Ze > 0$ surrounded by an electron “cloud”

$$\nabla^2 \Phi = \frac{1}{r} \partial_r^2 (r\Phi) \simeq \frac{4\pi e^2 n_0}{k_B T_e} \Phi \quad (r > 0)$$

$$\Phi(r) \xrightarrow{r \rightarrow 0} \frac{Ze}{r} \quad \longrightarrow \quad \Phi(r) = \frac{Ze}{r} e^{-r/\lambda_D}$$



Coulomb interactions between particles are screened at a distance $\lambda_D = (k_B T_e / 4\pi e^2 n_0)^{1/2}$

Model built on Debye-Hückel theory of electrolyte solutions

Debye screening and weak coupling

The Debye screening model requires $\frac{4\pi}{3}\lambda_D^3 n_e \gg 1$
i.e there are many electrons into the “Debye sphere”

Equivalent to: $\frac{e^2}{r_e} \ll k_B T_e$ $r_e = \left(\frac{3}{4\pi n_e}\right)^{1/3}$
 r_e : average distance between electrons ($(4\pi/3)r_e^3 n_e = 1$)

Binary interaction energy \ll kinetic energy
 (“weakly coupled” plasma)

Pinch effect and magnetic equilibrium

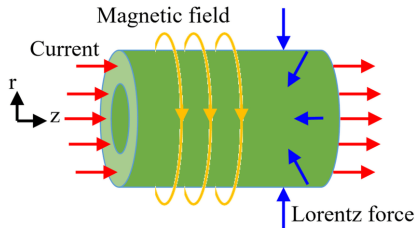
A beam of charged particles carrying a current density \mathbf{J} tends to shrink due to the self-generated magnetic field \mathbf{B}

The magnetic force can be compensated by the thermal pressure P

$$\frac{1}{c} \mathbf{J} \times \mathbf{B} - \nabla P = 0 \quad (1)$$

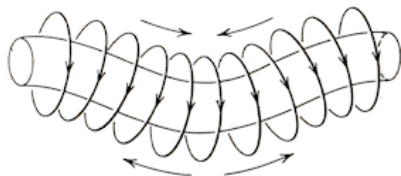
(Bennet pinch condition)

Eq.(1) is actually pivotal to all magnetic confinement schemes for fusion (Tokamak, Stellarators, Z-pinch)

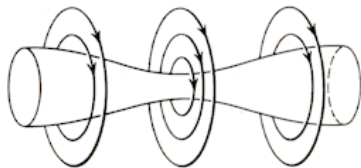


Pinch instabilities

The current-carrying plasma column is **unstable** with respect to bending and constriction



Kink instability



Sausage instability

Both instabilities are caused by the increased magnetic pressure (field lines become denser) at the locations of the perturbation

Figure from J.D. Jackson *Classical Electrodynamics* 2nd Ed. (Wiley, 1975)

Plasma instabilities . . .

List of plasma instabilities [edit]

- Buneman instability,^[3]
 - Farley–Buneman instability,^{[4][5]}
 - Jeans–Buneman instability,^{[6][7]}
 - Relativistic Buneman instability,^[8]
- Cherenkov instability,^[9]
- Coalescence instability,^[10]
 - Non-linear coalescence instability
- Chute instability,
- Collapse instability,
- Cyclotron instabilities, including:
 - Alfvén cyclotron instability
 - Cyclotron maser instability,^[11]
 - Electron cyclotron instability
 - Electrostatic ion cyclotron instability
 - Ion cyclotron instability
 - Magnetoacoustic cyclotron instability
 - Proton cyclotron instability
 - Non-resonant beam-type cyclotron instability
 - Relativistic ion cyclotron instability
 - Whistler cyclotron instability
- Dlocotron instability,^[12] (similar to the Kelvin-Helmholtz fluid instability).
- Disruptive instability (in tokamaks)^[13]
- Double emission instability,
 - Edge-localized modes,^{[14][15]}
 - Explosive instability (or Ballooning instability),^[16]
- Double plasma resonance instability,^[17]
- Drift instability^[18] (a.k.a. drift-wave instability,^[19] or universal instability,^[20]
- Firehose instability (a.k.a. hose instability), not to be confused with the similarly named Firehose instability in galactic dynamics
- Fish instability,
- Free electron maser instability,
- Gyrotron instability,
- Helical (Helix) instability,
- Jeans instability,^{[23][24]}
- Magnetic buoyancy instability
 - Interchange instability (a.k.a. flute instability),^[25]
 - Parker instability^[26] (a.k.a. undular instability or magnetic Rayleigh–Taylor instability)
 - Mixed instability (a.k.a. quasi-interchange instability)
- Magnetorotational instability (in accretion disks)
- Magnetothermal instability (Laser-plasmas),^[27]
- Modulational instability
- Non-Abelian instability,
- Pair instability (in supernovae)
- Peratt instability (stacked toroids)
- Pinch instability (a.k.a. Bennett pinch instability),^{[28][29]}
 - Sausage instability (m=0)
 - Kink instability (m=1)
 - Helical kink instability (a.k.a. helical instability)
- Rayleigh–Taylor instability (RTI, a.k.a. gravitational instability)
- Rotating instability,^[30]
- Tearing mode instability (or resistive tearing instability,^[31])
- Two-stream instability (a.k.a. beam-plasma instability, counter-streaming instability)
 - Beam acoustic instability

Kinetic equations

Distribution function $f_a = f_a(\mathbf{r}, \mathbf{p}, t)$: probability density in phase space cell $d^3\mathbf{r}d^3\mathbf{p}$ satisfying a continuity equation (assuming a constant number of particles):

$$\frac{df_a}{dt} = \frac{\partial f_a}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\dot{\mathbf{r}}_a f_a) + \frac{\partial}{\partial \mathbf{p}}(\dot{\mathbf{p}}_a f_a) = 0$$

$$\dot{\mathbf{r}}_a = \mathbf{v} = \frac{\mathbf{p}}{m_a} \quad \dot{\mathbf{p}}_a = q_a \left(\mathbf{E}(\mathbf{r}, t) + \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{r}, t) \right)$$

Mean field approach

- A distribution of point-like particles is equivalent to solve the equations of motion of all particles
→ a spatial average over a small volume Δ containing enough particles is assumed to yield a “smooth” distribution

$$f_a \longrightarrow \bar{f}_a \equiv \frac{1}{\Delta} \int_{\Delta} f_a d^3\mathbf{r}$$

- Fields are separated in averaged values + remainders i.e.

$$\mathbf{E} = \bar{\mathbf{E}} + \delta\mathbf{E} \quad \mathbf{B} = \bar{\mathbf{B}} + \delta\mathbf{B}$$

$\delta\mathbf{E}$ includes strong fluctuations of the field on a particle during a close encounter (“collision”) with another one

Mean field approach

Keep only average fields on the l.h.s of the equation and put the fluctuations on the r.h.s.:

$$\begin{aligned}\frac{d\bar{f}_a}{dt} &= \frac{\partial \bar{f}_a}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \bar{f}_a + q_a \left(\bar{\mathbf{E}} + \frac{\mathbf{v}}{c} \times \bar{\mathbf{B}} \right) \cdot \frac{\partial}{\partial \mathbf{p}} \bar{f}_a \\ &= \overline{-q_a \left(\delta \mathbf{E} + \frac{\mathbf{v}}{c} \times \delta \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{p}} f_a} = \left(\frac{\partial \bar{f}_a}{\partial t} \right)_{\text{col}}\end{aligned}$$

All binary interactions (“collisions”) are into $(\partial_t f_a)_{\text{col}}$

Mean field approach

Spatial averaging allows to define “smooth” charge and current densities from \bar{f}_a :

$$\bar{\rho}(\mathbf{r}, t) = \sum_a q_a \int \bar{f}_a(\mathbf{r}, \mathbf{p}, t) d^3\mathbf{p} \quad \bar{\mathbf{J}}(\mathbf{r}, t) = \sum_a q_a \int \mathbf{v} \bar{f}_a(\mathbf{r}, \mathbf{p}, t) d^3\mathbf{p}$$

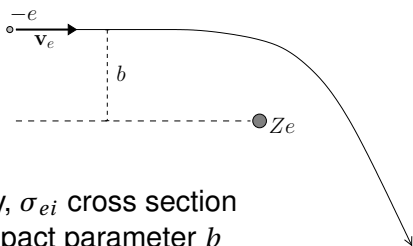
Self-consistent coupling with Maxwell equations:

$$\nabla \cdot \bar{\mathbf{E}} = 4\pi\bar{\rho} \quad \nabla \cdot \bar{\mathbf{B}} = 0 \quad \nabla \times \bar{\mathbf{E}} = -\frac{1}{c} \partial_t \bar{\mathbf{B}} \quad \nabla \times \bar{\mathbf{B}} = \frac{4\pi}{c} \bar{\mathbf{J}} + \frac{1}{c} \partial_t \bar{\mathbf{E}}$$

From now on we drop the average “-” symbol

Frequency of Coulomb collisions

Consider binary Coulomb interaction between an electron and an ion (“Rutherford” scattering)



$\nu_{ei} = n_e v_e \sigma_{ei}$ collision frequency, σ_{ei} cross section
Estimating the most relevant impact parameter b
from velocity variation $\Delta v_e \doteq v_e$ with

$$m_e \Delta v_e \approx F \Delta t \approx \frac{Ze^2}{b^2} \frac{b}{v_e} \quad \sigma_{ei} \approx \pi b^2 \approx \pi \frac{(Ze^2/m_e)^2}{v_e^4}$$

$$\nu_{ei} \propto \frac{n_e}{v_e^3} \propto \frac{n_e}{T_e^{3/2}} \quad \left(\frac{m_e v_e^2}{2} = T_e \right)$$

Coulomb logarithm

Exact calculation must deal with the classical logarithmic divergence of σ due to the long-range interaction $\propto 1/r^2$

$$\sigma_{ea} \approx \left(\frac{Ze^2}{m_e v_e^2} \right)^2 \ln \Lambda_{ea} \quad \Lambda_{ea} \equiv \frac{b_{\max}}{b_{\min}} \sim 10 - 20$$

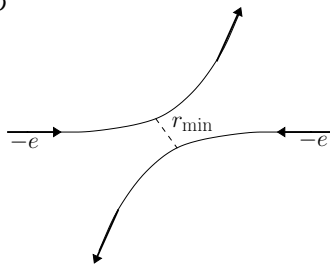
Screening at $r \approx \lambda_D$ determines $b_{\max} \approx \lambda_D$

$e - i$ scattering: $b_{\min} \approx \frac{\hbar}{m_e v_e}$

(quantum De Broglie length)

$e - e$ scattering: $b_{\min} = \max\left(r_{\min}, \frac{\hbar}{m_e v_e}\right)$

$\frac{e^2}{r_{\min}} = \frac{m_e v_e^2}{2}$ (energy conservation)



Collision vs. plasma frequencies

For both $e-i$ and $e-e$ scattering

$$\frac{\nu_{ea}}{\omega_p} \propto \frac{e^4 n_e (k_B T_e)^{-3/2} \ln \Lambda_{ea}}{(e^2 n_e / m_e)^{1/2}} \propto \frac{1}{n_e \lambda_D^3}$$

For $n_e \lambda_D^3 \gg 1$ i.e. $e^2 / r_e \ll k_B T_e$ the collective dynamics (timescale $\approx 1/\omega_p$) is faster than collisions (timescale $\approx 1/\nu_{ea}$)

The collisionless approximation is good for many (but not all) “hot” plasmas in space and laboratory

Vlasov equation

Neglect collisions completely:

$$\frac{df_a}{dt} = \frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} f_a + q_a \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{p}} f_a = 0$$

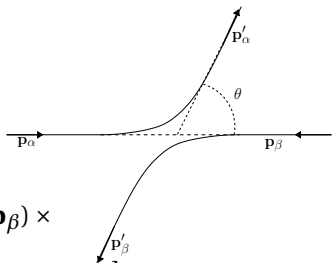
Repetita iuvant: \mathbf{E} and \mathbf{B} are the mean fields satisfying the self-consistent Maxwell equations

(in the electrostatic case the Vlasov-Maxwell system becomes the Vlasov-Poisson one with $\mathbf{E} = -\nabla\Phi$)

Note: in this limit there is no dissipative effect in the plasma (e.g. no electrical resistivity)

Collision term from statistical mechanics

Boltzmann collision integral for binary scattering between two previously uncorrelated particles α, β
(\mathbf{r}, t dependence omitted for brevity)



$$\left(\frac{\partial \bar{f}}{\partial t}\right)_{\text{col}}(\mathbf{p}_\alpha) = \int d^3\mathbf{p}_\beta \int d\Omega \frac{d}{d\Omega} \sigma_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}_\beta) \times \\ \times (\mathbf{v}_\beta - \mathbf{v}_\alpha) \left[f(\mathbf{p}'_\alpha) f(\mathbf{p}'_\beta) - f(\mathbf{p}_\alpha) f(\mathbf{p}_\beta) \right]$$

with $d\Omega = 2\pi \sin\theta d\theta$ and $\begin{cases} \mathbf{p}_\alpha + \mathbf{p}_\beta = \mathbf{p}'_\alpha + \mathbf{p}'_\beta \\ \mathcal{E}_\alpha + \mathcal{E}_\beta = \mathcal{E}'_\alpha + \mathcal{E}'_\beta \end{cases}$ ($\mathcal{E}_a = m_a v_a^2/2$)
(energy-momentum conservation)

BGK simple approximation to collision term

Idea: in the absence of other forces the $e - e$ collisions would thermalize the electrons so that f_e would relax to a Maxwellian

$$\left(\frac{\partial f_e}{\partial t}\right)_{\text{col}} = -\nu_{ee}(f_e - f_M)$$
$$f_M = f_M(\mathbf{p}) = \left(\frac{m_e}{2\pi k_B T_e}\right)^{3/2} e^{-\mathbf{p}^2/(m_e k_B T_e)}$$

(BGK = Bhatnagar, Gross, Krook)

Despite its simplicity this model can be used e.g. to derive basic expressions for the electrical and thermal conductivity of the plasma

Fluid description

Assuming the velocity distribution function is “under control” we can obtain equations for the density $n_a = n_a(\mathbf{r}, t)$ and average “fluid” velocity $\mathbf{u}_a = \mathbf{u}_a(\mathbf{r}, t)$:

$$n_a(\mathbf{r}, t) \equiv \int f_a(\mathbf{r}, \mathbf{p}, t) d^3 \mathbf{p} \quad \mathbf{u}_a(\mathbf{r}, t) \equiv \frac{1}{n_a} \int \mathbf{v} f_a(\mathbf{r}, \mathbf{p}, t) d^3 \mathbf{p}$$

Integrating the Vlasov equation over $d^3 \mathbf{p}$ yields the continuity equation

$$\partial_t n_a + \nabla \cdot (n_a \mathbf{u}_a) = 0$$

Reminder: $\rho = \sum_a q_a n_a \quad \mathbf{J} = \sum_a q_a n_a \mathbf{u}_a$

Fluid description

Multiply the Vlasov equation by \mathbf{v} and integrate again over $d^3\mathbf{p}$:

$$m_a n_a (\partial_t + \mathbf{u}_a \cdot \nabla) \mathbf{u}_a = q_a n_a (\mathbf{E} + \mathbf{u}_a \times \mathbf{B} / c) - \nabla P_a$$

P_a : pressure term (assumed as isotropic)

$$P_a = P_a(\mathbf{r}, t) \equiv \int (\mathbf{v} - \mathbf{u}_a)^2 f(\mathbf{r}, \mathbf{p}, t) d^3\mathbf{p}$$

$(\mathbf{v} - \mathbf{u}_a$: “random” or thermal part of the velocity)

To close the hierarchy of fluid equations an equation of state for P_a can be assumed e.g. $P_a = n_a k_B T_a$

High frequency waves in a plasma

$m_i \gg m_e \rightarrow$ assume mobile electrons and ions at rest
General wave equation for \mathbf{E} from Maxwell equations:

$$\left(\nabla^2 - \frac{1}{c^2} \partial_t^2\right) \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \frac{4\pi}{c^2} \partial_t \mathbf{J}$$

Assume monochromatic fields i.e. $\mathbf{E}(\mathbf{r}, t) = \text{Re} [\tilde{\mathbf{E}}(\mathbf{r}) e^{-i\omega t}]$
Linearize the fluid equations for electrons neglecting P_e for the moment (“cold” plasma)

$$\partial_t \mathbf{u}_e = -\frac{e}{m_e} \mathbf{E} \qquad \tilde{\mathbf{J}} = -en_e \tilde{\mathbf{u}}_e = -\frac{i}{4\pi} \frac{\omega_p^2}{\omega} \tilde{\mathbf{E}}$$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{\mathbf{E}} - \nabla(\nabla \cdot \tilde{\mathbf{E}}) = \frac{\omega_p^2}{c^2} \tilde{\mathbf{E}} \qquad \text{Helmoltz equation}$$

Transverse electromagnetic (EM) waves

Taking $\nabla \cdot \mathbf{E} = 0$ and introducing $\varepsilon(\omega) = n^2(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\left(\nabla^2 + \varepsilon(\omega) \frac{\omega^2}{c^2} \right) \tilde{\mathbf{E}} = \left(\nabla^2 + n^2(\omega) \frac{\omega^2}{c^2} \right) \tilde{\mathbf{E}} = 0$$

$\varepsilon(\omega)$ dielectric function, $n(\omega)$ refractive index

Plane waves $\tilde{\mathbf{E}}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r}}$ $\mathbf{k} \cdot \mathbf{E}_0 = 0$ $\mathbf{B} = \mathbf{k} \times \mathbf{E} / k$

→ dispersion relation $k^2 c^2 = \varepsilon(\omega) \omega^2 = \omega^2 - \omega_p^2$

Phase velocity $v_p = \frac{\omega}{k} = \frac{c}{n(\omega)} > c$, group velocity $v_g = \partial_k \omega = \frac{c^2}{v_p} < c$

Propagation requires a real value of k i.e.

$$k^2 > 0 \quad \leftrightarrow \quad \varepsilon(\omega) > 0 \quad \leftrightarrow \quad \omega > \omega_p$$

Cut-off or “critical” density for EM waves

$$\omega > \omega_p \quad \leftrightarrow \quad n_e < n_c \equiv \frac{m_e \omega^2}{4\pi e^2} = 1.1 \times 10^{21} \text{ cm}^{-3} \left(\frac{\lambda}{1 \mu\text{m}} \right)^{-2}$$

$n_e < n_c$: “underdense” transparent plasma

$n_e > n_c$: “overdense” opaque plasma

Metals ($n_e \approx 10^{23} \text{ cm}^{-3}$) reflect visible light ($\lambda = 0.4 - 0.8 \mu\text{m}$)

Ionosphere ($n_e \approx 10^4 - 10^6 \text{ cm}^{-3}$) reflects radio waves ($\lambda \gtrsim 1 \text{ m}$)

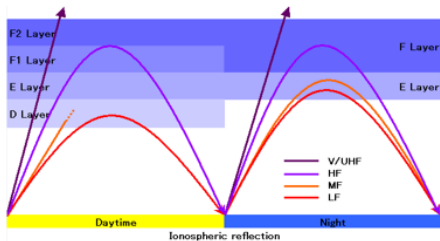
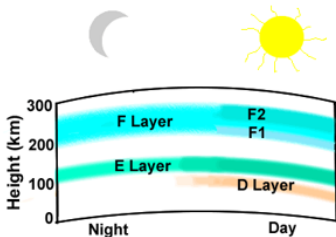


Figure from engineerstutor.com

Longitudinal electrostatic (ES) waves

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{E}} - \nabla(\nabla \cdot \tilde{\mathbf{E}}) = \frac{\omega_p^2}{c^2}\tilde{\mathbf{E}} \quad \text{Helmoltz equation}$$

$$\text{Take } \tilde{\mathbf{E}}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{k} \parallel \mathbf{E}_0, \quad \nabla \cdot \mathbf{E} = \mathbf{k} \cdot \mathbf{E}, \quad \mathbf{B} = 0$$

$$\nabla(\nabla \cdot \tilde{\mathbf{E}}) = \nabla^2 \tilde{\mathbf{E}} \quad \Rightarrow \quad (\omega^2 - \omega_p^2)\tilde{\mathbf{E}} = 0 \quad \Rightarrow \quad \omega = \omega_p$$

(Phase velocity is arbitrary, group velocity is zero)

→ “cold” non-propagating plasma oscillations recovered

Warning: the “cold” plasma assumption holds only if

$$u_{\text{osc}} = \frac{eE_0}{m_e \omega_p} \gg v_{\text{th}} = \left(\frac{k_B T_e}{m_e}\right)^{1/2}$$

Langmuir ES waves in warm plasma

Linearized equation with pressure term included:

$$\partial_t \mathbf{u}_e = -\frac{e}{m_e} \mathbf{E} - \frac{\nabla P_e}{m_e n_e}$$

Assume $P = \gamma n_e k_B T_e$ and $\gamma = 3$ for 1D adiabatic compression
Couple with Gauss law & linearized continuity equation

$$\nabla \cdot \mathbf{E} = 4\pi e(n_e - n_0) \quad \partial_t n_e + n_0 \nabla \cdot \mathbf{u}_e \simeq 0$$

$$\longrightarrow \quad \omega^2 = \omega_p^2 + 3k^2 v_{th}^2 \quad \text{dispersion relation}$$

Plasma oscillations become propagating thanks to pressure

Note: the pressure term has no effect on transverse EM waves

Kinetic theory of ES waves

Linearizing the 1D Vlasov-Poisson system ($f_e \simeq f_0(v) + f_1(x, v, t)$) and searching for harmonic ES plane wave solutions yield the dispersion relation $\omega = \omega(k)$ in implicit form

$$1 = \frac{\omega_p^2}{k^2} \int_{-\infty}^{+\infty} \frac{\partial_v f_0(v)}{v - \omega/k} dv$$

The **singularity** at $v = \omega/k = v_p$ indicates a “resonant” interaction between the wave and the particles moving at the phase velocity v_p

Proper approach to the problem by Landau yields an **imaginary** part of the frequency:

$$\omega \simeq (\omega_p^2 + 3k^2 v_{th}^2)^{1/2} + i\omega_I \quad \omega_I \simeq \frac{\pi}{2} \frac{\omega_p^3}{k^2} \left(\frac{\partial f_0(v)}{\partial v} \right)_{v=v_p=\omega/k}$$

Landau damping

$$\text{If } \left(\frac{\partial f_0(v)}{\partial v} \right)_{v=v_p=\omega/k} < 0 \text{ then } \omega_I < 0$$

(case for a Maxwellian or bell-shaped distribution)

→ the ES wave is **damped** in time despite the absence of collisions.

This is a **reversible** process of energy exchange between the wave and the particles that has been verified experimentally.

Mathematician Cedric Villani who defined the full Boltzmann equation “the most beautiful equation ever” later obtained the Fields Medal for his study of Landau damping in the fully nonlinear Vlasov-Poisson system



Ion acoustic (“sound”) waves

On the time scale of ion motion $\sim \frac{1}{\omega_{pi}} = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{1}{\omega_{pe}}$ electrons can be assumed in Boltzmann equilibrium $n_e = n_0 \exp(\Phi/k_B T_e)$
Linearized equations for ions ($\mathbf{E} = -\nabla\Phi$):

$$\partial_t \mathbf{u}_i = -\frac{Ze}{m_i} \nabla\Phi - \frac{\nabla P_i}{m_i n_i}$$

$$\nabla^2 \Phi = 4\pi e(n_e - n_i) \quad \partial_t n_i + n_0 \nabla \cdot \mathbf{u}_i \simeq 0$$

For low frequencies $\omega \simeq kc_s$ with $c_s = \left(\frac{k_B(ZT_e + T_i)}{m_i}\right)^{1/2}$

Note: ion-acoustic waves exist also for “cold” ions ($T_i = 0$) and can be Landau-damped like Langmuir waves

Waves in a magnetized plasma

Many plasma of interests (solar wind, ionosphere, fusion machines . . .) are immersed in strong magnetic fields (\mathbf{B}_0)

- particles tend to orbit around \mathbf{B}_0
at the *cyclotron frequency*

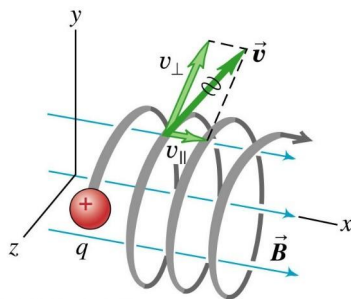
$$\omega_{ca} = \frac{q_a B_0}{m_a c}$$

→ resonances at $\omega = \omega_{ca}$

- \mathbf{B}_0 breaks chiral invariance

→ plasma becomes *anisotropic*
and *birefringent*

(wave propagation depends on
polarization)

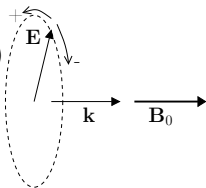


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Propagation along the magnetic field

Refractive index for left- (+) and right-handed (-) circular polarizations

$$n_{\pm}^2(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_c)}$$



Resonance at $\omega = \omega_c$ *only* for “+” polarization:

$\mathbf{F} = -e\mathbf{E}$ must be parallel to the gyration velocity

At very low frequencies only “-” polarization propagates with

phase velocity $v_{p-} \simeq c \frac{(\omega\omega_c)^{1/2}}{\omega_p}$: dispersive “whistler” waves

(excited by lightnings in the ionosphere)

Magnetohydrodynamics (MHD)

Single fluid approach for a collisional plasma in a regime of low frequencies, strong \mathbf{B} and negligible charge separation

$$\partial_t \rho_m + \nabla \cdot \rho_m \mathbf{u} = 0 \quad (\rho_m = m_i n_i + m_e n_e \simeq m_i n_i)$$

$$\rho_m \frac{d\mathbf{u}}{dt} = \frac{1}{c} \mathbf{J} \times \mathbf{B} - \nabla P \quad (P = P_e + P_i)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}$$

$$\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} = \eta \mathbf{J} \stackrel{\eta \rightarrow 0}{\simeq} 0$$

$$\partial_t \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) = \eta \frac{c}{4\pi} \nabla^2 \mathbf{B} \stackrel{\eta \rightarrow 0}{\simeq} 0$$

If resistivity $\eta \rightarrow 0$ (“ideal” MHD) the field lines of \mathbf{B}' are “frozen” in the fluid

MHD waves

Longitudinal “magnetosonic” and Transverse “Alfvén” waves

$$v_a \equiv B_0 / (4\pi m_i n_0)^{1/2}$$

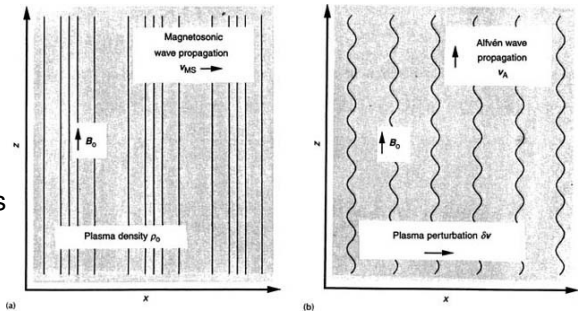
(Alfvén velocity)

MS \parallel wave:

$$\omega = k(c_s^2 + v_A^2)^{1/2}$$

(magnetic pressure adds up to thermal pressure)

A \perp wave: $\omega = k v_a$



Simpson et al, “Alfvén, Hannes Olof Gösta (1908-1995)”, *Encyclopedia of Planetary Science* (Springer, 1997)

H. Alfvén: Nobel prize, Physics (1970)

This was theory ...

... and in practice?

What's next for plasma applications?

Which research might be awarded in the (near) future?

- ▶ fusion?
- ▶ laser-plasma accelerators?
- ▶ “green” plasma discharge technologies?
- ▶ aerospace?
- ▶ ...
- ▶ plasma propulsion? (My best wishes to you!)

Look for the future into the plasma sphere!

(another example of
discharge plasma
that works!)

Thank you for your attention

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