Magnetic pressure in a solenoid

A constant current intensity I, driven by an ideal current generator, flows in the coils of a cylindrical solenoid having radius a, length $h \ll a$ and n coils per unit length. Boundary effects are negligible.
a) By directly computing the magnetic force on the coils show that on the surface of the solenoid there is a pressure $P = B_0^2/2\mu_0$, with $B_0 = \mu_0 n I$ the inner field.
b) Now find the variation of the magnetic energy for a small increase Δa of the radius of the solenoid, and the corresponding work done by the current generator in order to keep I constant. Use the results to derive the pressure P again.

Solution

a) The magnetic force on an infinitesimal segment of a circuit having length $d\mathbf{l}$ is

$$d\mathbf{f} = I d\mathbf{l} \times \mathbf{B} \; . \tag{1}$$

Being $\mathbf{B} = B\hat{\mathbf{z}}$, the force $d\mathbf{f}$ is perpendicular to the surface of the solenoid and directed outwards, i.e. the solenoid tends to expand radially. Since $\mathbf{B}$ is discontinuous in the idealized limit of a surface current layer of infinitesimal thickness, we take as the value of $\mathbf{B}$ on the surface the average $B_0/2$ between the inner field $B(r < a) = B_0$ and the outer field $B(r > a) = 0$, in the limit of infinite length $a/h \to 0$. Thus, $df = IB_0 dl/2$. To find the force on a surface element of area $dS = dhdl$ we notice that it contains ndh circuit segments, thus $dF = (IB_0 dl/2)ndh = (nIB_0/2)dldh = (B_0^2/2\mu_0)dS$ from which we eventually obtain $P = dF/dS = B_0^2/2\mu_0$.

Alternatively, pursuing a more formal approach we may use $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$ to write the force per unit volume as $\mathbf{J} \times \mathbf{B} = -\hat{\mathbf{r}}\,(\partial_r B_z)\, B_z/\mu_0$. Thus, by integrating over the thickness of the current layer

$$P = \int \mathbf{J} \times \mathbf{B} \cdot \hat{\mathbf{r}} dr = -\frac{1}{2\mu_0} \int_{a^-}^{a^+} \partial_r B_z^2 dr = \frac{B_0^2}{2\mu_0} \; . \tag{2}$$

b) The magnetic energy of the solenoid may be written using the "energy density" associated to the magnetic field, $u_m = B^2/2\mu_0$. Being $B = B_0$ inside and $B = 0$ out of the solenoid, we obtain $U_m = (B_0^2/2\mu_0)\pi a^2 h$. Thus, for a small variation of the radius $\Delta a > 0$ the energy increases by $\Delta U_m = (B_0^2/\mu_0)\pi h a \Delta a > 0$.

The variation of a while keeping the current intensity I constant leads to a variation of the magnetic flux Φ through the solenoid, thus an induced electromotive force $\mathcal{E} = -d\Phi/dt$ appears. To keep I constant the generator must provide the power $P = -\mathcal{E}I$. The total work done is thus

$$W = \int P dt = I \int d\Phi = I\Delta\Phi \; , \tag{3}$$

where the total variation of the flux is $\Delta\Phi = \Delta(B_0 \pi a^2 n h) = 2\pi B_0 h n a \Delta a$. In order to provide such work the internal energy of the generator decreases by $\Delta U_g = -W = -2\pi n I B_0 h a \Delta a = -2\Delta U_m$. Thus, the energy of the whole system (solenoid plus generator) varies by $\Delta U_{\text{tot}} = \Delta U_g + \Delta U_m = -\Delta U_m < 0$, so that $-\Delta U_{\text{tot}}/\Delta a > 0$ confirming that the solenoid tends to expand radially driven by a pressure $P = -(\Delta U_{\text{tot}}/\Delta a)/(2\pi a h) = B_0^2/2\mu_0$.