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Elements for a theory of financial risks

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Abstract

Estimating and controlling large risks has become one of the main concerns of financial institutions. This requires the development of adequate statistical models and theoretical tools (which go beyond the traditional theories based on Gaussian statistics), and their practical implementation. Here we describe two interrelated aspects of this program: we first give a brief survey of the peculiar statistical properties of the empirical price fluctuations. We then review how an option pricing theory consistent with these statistical features can be constructed, and compared with real market prices for options.

Keywords: Financial analysis; Scaling; Power-law correlations

1. Introduction

The efficiency of the theoretical tools devised to address the problems of risk control, portfolio selection and derivative pricing strongly depends on the adequacy of the stochastic model chosen to describe the market fluctuations. Historically, the idea that price changes could be modelled as a Brownian motion dates back to Bachelier [1]. This hypothesis, or some of its variants (such as the geometrical Brownian motion, where the log of the price is a Brownian motion) is at the root of most of the modern results of mathematical finance, with Markowitz portfolio analysis, the capital asset pricing model (CAPM) [2] and the Black–Scholes formula [3] standing out as paradigms. The reason for success is mainly due to the impressive mathematical and probabilistic apparatus available to deal with Brownian motion problems, in particular

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Ito's stochastic calculus. An important justification of the Brownian motion description lies in the central limit theorem (CLT), stating that the sum of N identically distributed, weakly dependent random changes is, for large N, a Gaussian variable. In physics or in finance, the number of elementary changes observed during a time interval t is given by $N = t/\tau^*$, where τ^* is an elementary correlation time, below which changes of velocity (for the case of a Brownian particle) or changes of 'trend' (in the case of the stock prices) cannot occur. The use of the CLT to substantiate the use of Gaussian statistics in any case requires that $t \gg \tau^*$. In financial markets, τ^* turns out to be of the order of several minutes, which is not that small compared to the relevant time scales (days), in particular when one has to worry about the *tails* of the distribution (corresponding to large shocks) which sometimes disappear only very slowly. The fact that the tails of the distribution of returns are much 'fatter' than predicted by the Gaussian is well known, in particular since the seminal work of Mandelbrot [4,5], where the idea that price changes are still independent, but distributed according to a Lévy stable law was first proposed. This model however fails in two respects. First, the tails of the distribution of returns is now much *overestimated*; in particular, the variance appears to be well defined in most financial markets, while it is infinite for Lévy distributions. Second, and perhaps more importantly in view of its application to option markets, the amplitude of the fluctuations (measured, say, as the local variance) appears to be itself a randomly fluctuating variable with rather long-range correlations.

The aim of these lectures is to provide a short survey of the most prominent statistical properties of the fluctuations of rather liquid markets, which are characterized by what one could call 'moderate' fluctuations (for a review, see e.g. Refs. [6–8]). Less liquid markets sometimes behave rather differently and more 'extreme' fluctuations can be observed. We shall then present a theory for option pricing and hedging in the general case where the underlying stock fluctuations are not Gaussian. In this case, perfect hedging is in general impossible, but optimal strategies can be found (analytically or numerically) and the associated residual risk can be estimated. We show that the volatility 'smile' observed on option markets can be understood using a cumulant expansion, and discuss the idea of an implied 'kurtosis', which is (on liquid markets) very close to the actual (maturity dependent) kurtosis of the historical data.

2. A short survey of empirical data

2.1. Linear correlations

We shall denote in the following the value of the stock (or any other asset) at time t as $x(t)$, and the variation of the stock on a given time interval τ as $\delta_\tau x(t) = x(t+\tau) - x(t)$. The time delay τ can be as small as a few seconds in actively traded markets. However, on these short time scales, the fluctuations cannot be considered to be independent. For

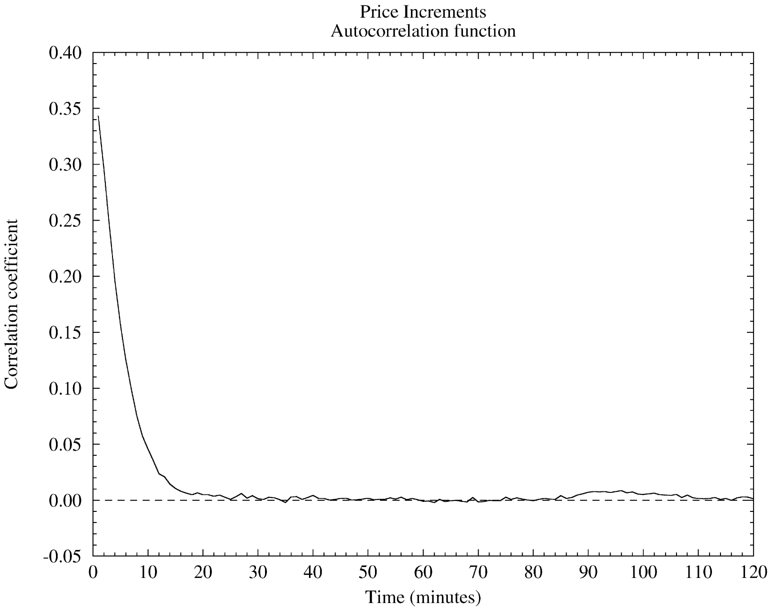

Fig. 1. Correlation function $C(\Delta t)$ of the 1 min price increments of the S& P 500, for the period 1983–1996, which decays to zero on the scale of $\tau^* \simeq 15$ min. Note that τ^* has drifted to smaller values with time.

example, the second-order correlation function defined as [1]

$$C(\Delta t) = \frac{\langle \delta_\tau x(t+\Delta t)\delta_\tau x(t)\rangle}{\langle \delta_\tau x(t)^2\rangle} \tag{1}$$

is significantly non-zero up to $\Delta t = \tau^* = 15$ min, as shown in Fig. 1.

Therefore, it is certainly inadequate to think of the price process as a 'martingale' for small time increments. This however does not necessarily mean that there are arbitrage opportunities: one can easily see that even very small transaction costs prevent the use of these short time correlations, which would imply a very high (and thus very costly) trading frequency [9]. For time delays Δt longer than a certain τ^*, say 1 h, the autocorrelation of price increments is very nearly zero; correspondingly, the variance of price increments grow linearly with Δt for $\Delta t > \tau^*$.

2.2. Distribution of elementary increments

The simplest idea is thus that the increments [2] $\delta(t)$ are independent identically distributed (IID) random variables. In this case, the knowledge of the distribution

[1] In principle, the average value of $\delta_\tau x(t)$ should be removed, but this leads to insignificant corrections on the time scales considered. For the same reason, we neglect the difference between the variation of the price and the more often studied variation of the log of the price.

[2] In the following, we shall simplify the notation and use $\delta(t) \equiv \delta_{\tau^*}x(t)$ for the increments on the time scale τ^*.

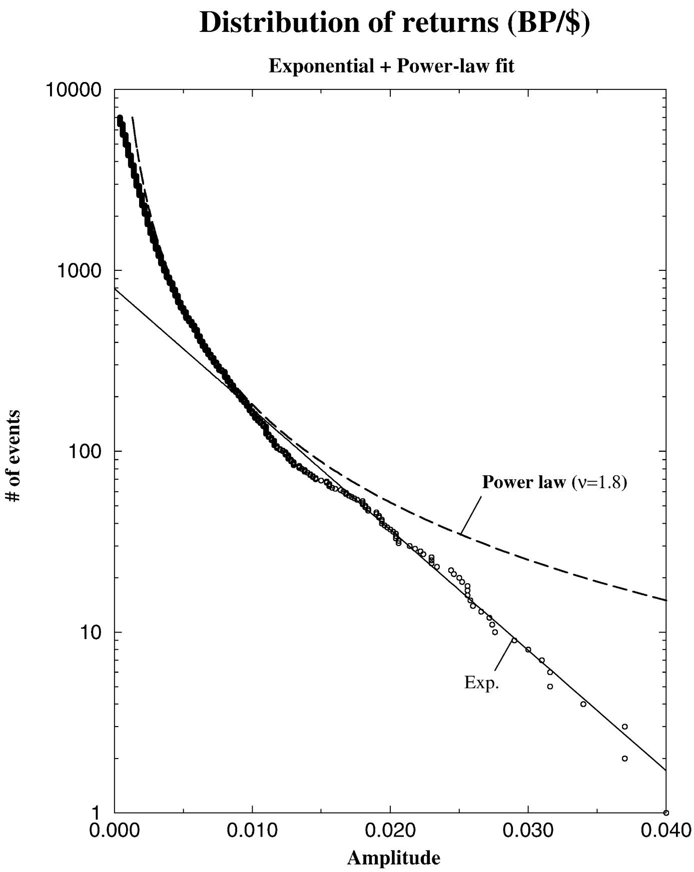

Fig. 2. Cumulative distribution of the British Pound/US \$ negative increments for $\tau^* = 30$ min, in the period 1991–1995. The 'near-tail' of the distribution can be fitted by a power law, which clearly overestimates the 'far-tail'. The latter is well represented by an exponential fall-off, Eq. (2): note the linear-log scale, although a high power law is difficult to distinguish from an exponential without much more data points [12,13].

density P^* of δ would suffice to reconstruct the distribution of increments on any time delay Δt larger than τ^*, through a simple convolution. The shape of P^* is strongly non-Gaussian: an estimate of its kurtosis κ_{τ^*} on a historical basis leads to numbers on the order of 20 (again on liquid markets). As shown in Fig. 2 in the example of the British Pound/U.S. \$ time series, the tails of P^* decay as an exponential

$$P(\delta) \propto \exp(-\lambda_\pm |\delta|)\,, \quad \delta \to \pm\infty\,. \tag{2}$$

Note however that several authors have reported a somewhat slower decay, as a power law $|\delta|^{-1-\nu}$ with a rather large exponent $\nu \simeq 3$–4 [10–13], or possibly a 'stretched exponential' [14]. Such slowly decaying tails survive upon convolution, and are thus particularly relevant for extreme risks forecasts.

A reasonable fit of P^* on most markets can be achieved using a symmetrical, 'truncated' Lévy distribution [9,15,16], defined in Fourier space as

$$\log P^*(z) = \frac{A}{\cos \pi\mu/2}[\lambda^\mu - (\lambda^2 + z^2)^{\mu/2} f(z)] ,$$

$$f(z) = \cos\left(\mu \arctan \frac{|z|}{\lambda}\right) , \tag{3}$$

where A is the scale factor, μ is the 'tail exponent', which is found to be close to 1.5 for all markets, and λ describes the exponential fall of the far-tails. Note that when $\mu = 2$, one recovers the Gaussian distribution, while for $\lambda \to 0$, one finds the stable Lévy distributions proposed by Mandelbrot [4,5], with tails which decay as $|\delta|^{-1-\mu}$. Note that Student distributions also do a good job in fitting the data [9].

2.3. Anomalous decay of the kurtosis and volatility persistence

The Nth autoconvolution of P^*, where $N = \Delta t/\tau^*$ is simply obtained by scaling the parameter A to NA. With no further adjustable parameters, this leads to a fair representation of the historical distribution of $\delta_{\Delta t} x$ [9,17]. However, some systematic differences (in particular in the tails of the distribution) show up, which reveal the inadequacy of such a simple IID hypothesis. For example, it is easy to show that under such an hypothesis, the kurtosis of the increments on scale $\Delta t = N\tau^*$ should be a factor N smaller than the kurtosis on scale τ^*. Empirically, however, one finds a much slower decay, as $\kappa_N = \kappa_1/N^\lambda$, with λ in the range 0.3–0.6 (see Fig. 3). This can be related to another observation, which was made many times in Refs. [6–8,18–20]: higher-order correlation functions of the price increments, such as

$$C_2(m) = \frac{\langle \delta^2(t + m\tau^*)\delta^2(t)\rangle}{\langle \delta^2(t)\rangle^2} - 1 \tag{4}$$

decay only very slowly with time. A simple fit of this decay is again of the power-law type [6–8,21,22]

$$C_2(m) \simeq \frac{C_0}{m^{\lambda'}} \tag{5}$$

with an exponent λ' in the same range as λ. A simple way to rationalize these findings is to assume that while the *sign* of the price increment is completely decorrelated as soon as $\Delta t > \tau^*$, the amplitude of the increment (which is a measure of the market activity) is correlated in time. Bursts of market activity, related to external news or crisis, can easily persist for several days, sometimes months – this explains why $C_2(m)$ decays rather slowly. We shall thus assume that the increment δ can be written as

$$\delta(t_i) = \gamma(t_i) \times \varepsilon(t_i), \quad t_i = i\tau^* , \tag{6}$$

where the scale $\gamma > 0$ measures the amplitude of the increment, which can be thought of as the local volatility of the market. The random variable ε is short-range correlated (over time τ^*) of mean zero and variance unity (and independent from γ). Note that ε

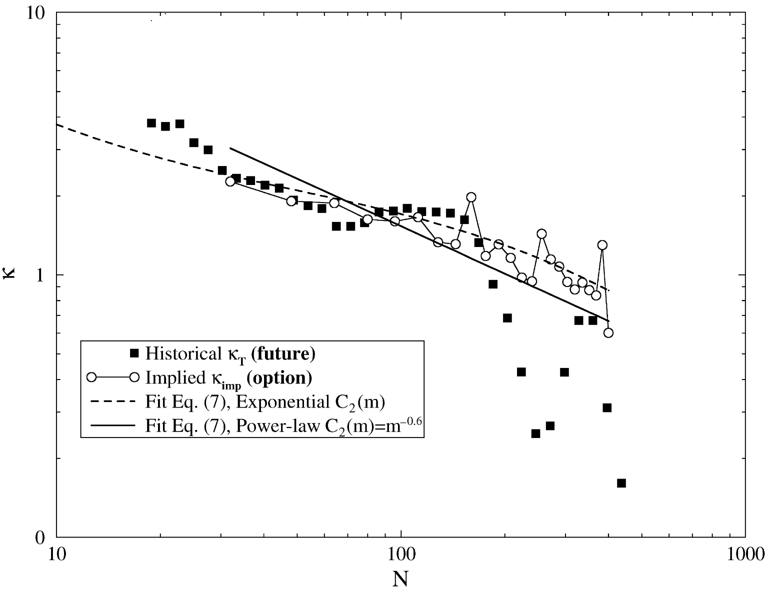

Fig. 3. Plot (in log–log coordinates) of the average implied kurtosis κ_{imp} (determined by fitting the implied volatility for a fixed maturity by a parabola) and of the empirical kurtosis κ_N (determined directly from the historical movements of the BUND contract), as a function of the reduced time scale $N = T/\tau$, $\tau = 30$ min. All transactions of options on the BUND future from 1993 to 1995 were analysed along with 5 min tick data of the BUND future for the same period. We show for comparison a fit with $\kappa_N \simeq N^{-0.6}$ (dark line). A fit with an exponentially decaying $C_2(m)$ is also however acceptable (dotted line).

is not necessarily Gaussian, as assumed in ARCH-like models [18–20]. It is then rather easy to show that the kurtosis of the increments on scale $\Delta t = N\tau^*$ can be expressed as [9,23]

$$\kappa_N = \frac{1}{N}\left[\kappa_0 + (3+\kappa_0)C_2(0) + 6\sum_{m=1}^{N}\left(1-\frac{m}{N}\right)C_2(m)\right], \tag{7}$$

where κ_0 is the kurtosis of the random variable ε. Note that even if $\kappa_0 = 0$, the kurtosis of P^* is non-zero due to the randomly fluctuating scale parameter γ. If one furthermore assumes that C_2 decays as a power law (Eq. (5)), then from Eq. (7), κ_N decays for large N as $N^{-\lambda}$ with $\lambda = \lambda'$.

2.4. Apparent (multi)-scaling behaviour

Hence, the fact that the scale γ of the random increments has long-range temporal correlations induces an anomalously slow convergence of the sum $x(N\tau^*) - x(0) = \sum_{i=1}^{N} \delta(t_i)$ towards the Gaussian distribution. One should note that this can induce *apparent* scaling behaviour on restricted time intervals. For example, a numerical simulation of the random scale model (6) with a slowly decaying $C_2(m)$ can be analysed in scaling terms [24], i.e., fitting the moments of $x(N\tau^*) - x(0)$ as power laws:

$$\langle |x(N\tau^*) - x(0)|^q \rangle \propto N^{\zeta_q} \, . \tag{8}$$

This actually works quite well, and leads to a family of exponents ζ_q which deviates from the theoretical straight line $\zeta_q = q/2$ which holds in the limit $N \to \infty$, provided all the moments of γ exist. For $q=2$, the relation $\zeta_2 = 1$ holds because the linear correlation function $C(\Delta t)$ is zero for $\Delta t > \tau^*$. For $q = 4$ one finds, using the definition of the kurtosis

$$\langle [x(N\tau^*) - x(0)]^4 \rangle = (3 + \kappa_N) \langle [x(N\tau^*) - x(0)]^2 \rangle^2 \propto 3N^2 + \kappa_1 N^{2-\lambda} \, , \tag{9}$$

which is thus the sum of two power laws. This can be however fitted with a unique 'effective' exponent ζ_4, which can be substantially below the asymptotic value $\zeta_4^\infty = 2$ since $N^{-\lambda}$ is not very small in practice. This argument holds true for higher moments for which $\zeta_q < q/2$; the difference between ζ_q and $q/2$ actually grows with q (for a fixed value of N) [24]. We believe that this might be the reason for the 'multifractal' scaling recently reported in the Refs. [25,26].

2.5. Time-reversal symmetry

The last empirical fact which we would like to comment on is the question of time-reversal symmetry. Is it possible to detect the direction of time in a financial time series? The answer to this question would be no for a simple Brownian motion, for example, but also for a much wider class of processes, such as the multifractal time construction proposed in Ref. [26]. Correlation functions sensitive to the arrow of time have been proposed by Pomeau [27]. One example is

$$C_T(\Delta t) = \langle x(t) \, [x(t + \Delta t) - x(t + 2\Delta t)] \, x(t + 3\Delta t) \rangle \, , \tag{10}$$

which is non-zero if the time triplets $t, t + \Delta t, t + 3\Delta t$ and $t, t + 2\Delta t, t + 3\Delta t$ cannot be distinguished statistically. For the price series itself, the above correlation function is zero within error bars. But when one studies the 'volatility' process γ, then this skew correlation function is distinctively non-zero, and reaches a maximum (in the case of the S& P 500) for $\Delta t \simeq 1$ month. This shows that the volatility time series is *not* invariant under time reversal. A similar conclusion [3] is also reported in Ref. [22], where the authors observe that a high 'coarse-grained' volatility in the past increases in a causal way today's 'fine-grained' volatility. This is not unreasonable, as one feels intuitively that an anomalously large change of the close to close price over – say – the previous week triggers more intraday activity the following week. In any case, we feel that this absence of time reversal symmetry is an important (albeit less frequently discussed) stylized fact of financial time series.

[3] See the work of M. Dacorogna et al. on 'HARCH' models, work available at http://www.olsen.ch.

3. Implications for option pricing

3.1. General framework

We now turn to the problem of option pricing and hedging when the statistics for price increments have the non-Gaussian properties discussed above. The distinctive feature of the continuous time random walk model usually considered in the theory of option pricing is the possibility of perfect hedging (for a remarkable introduction, see Ref. [28]), that is, a complete elimination of the risk associated to option trading [3]. This property however no longer holds for more realistic models [29].

Let us write down the global wealth balance $\Delta W|_0^T$ associated with the writing of a 'call' option of maturity T and exercise price x_s [9]:

$$\Delta W|_0^T = \mathscr{C}(x_0, x_s, T)\exp(rT) - \max(x(T) - x_s, 0) \tag{11}$$

$$+ \sum_i \phi(x, t_i)\exp(r(T - t_i))[\delta_i - rx(t_i)\tau]\,, \tag{12}$$

where $\mathscr{C}(x_0, x_s, T)$ is the price of the call, $x_0 = x(t=0)$ and $\phi(x,t)$ the trading strategy, i.e., the number of stocks per option in the portfolio of the option writer. Finally, r the (constant) interest rate. The second term defines the option contract: the profit of the buyer of the option is equal to $x_s - x(T)$ if $x(T) > x_s$ (i.e., if the option is exercised) and zero otherwise: the option is an insurance contract which guarantees to its owner a maximum price for acquiring a certain stock at time T. Conversely, a 'put' option would guarantee a certain minimum price for the stock held by the owner of the option.

A natural procedure to fix the price of the option $\mathscr{C}(x_0, x_s, T)$ and the optimal strategy $\phi^*(x,t)$ was proposed independently in Refs. [29–31] and further discussed in Refs. [9,32–36]. It consists of imposing a *fair game condition*, i.e.,

$$\langle \Delta W|_0^T[\phi]\rangle = 0 \tag{13}$$

and a *risk minimization condition*

$$\left.\frac{\delta\langle \Delta W|_0^T[\phi]^2\rangle}{\delta\phi(x,t)}\right|_{\phi^*} = 0\,. \tag{14}$$

Here, we assume that the variance of the wealth variation is a relevant measure of the risk. However, other measures are possible, such as higher moments of the distribution of ΔW, or the 'Value-at-Risk', which is directly related to the weight contained in the negative tails of the distribution of $\Delta W|_0^T$.

The notation $\langle\ldots\rangle$ in Eqs. (13) and (14) means that one averages over the probability of the different trajectories. The explicit solution of Eqs. (13) and (14) for a general uncorrelated process (i.e., $\langle\delta_i\delta_j\rangle = 0$ for $i \neq j$) is relatively easy to write if the average bias $\langle\delta\rangle$ and the interest rate r are negligible [4] which is the case for short maturities T. In this case, one finds [29]

$$\mathscr{C}(x_0, x_s, T) = \int_{x_s}^{\infty} \mathrm{d}x'(x' - x_s)P(x', T|x_0, 0)\,, \tag{15}$$

$$\phi^*(x,t) = \int_{x_s}^{\infty} \mathrm{d}x' \langle\delta\rangle_{(x,t)\to(x',T)} \frac{(x' - x_s)}{\sigma^2(x,t)} P(x', T|x, t)\,, \tag{16}$$

where $\sigma^2(x,t) = \langle\delta^2\rangle|_{x,t}$ is the 'local volatility' – which may depend on x, t – and $\langle\delta\rangle_{(x,t)\to(x',T)}$ is the mean instantaneous increment conditioned to the initial condition (x,t) and a final condition (x', T). The *minimal* residual risk, defined as $\mathscr{R}^* = \langle \Delta W|_0^T[\phi^*]^2\rangle$ is in general strictly positive (and in practice rather large), except for Gaussian fluctuations *in the continuous limit*, where the residual risk is strictly zero. In this limit, the above equations (13) and (14) actually exactly lead to the celebrated Black–Scholes option pricing formula. In particular, one can check that ϕ^* is related to $\mathscr{C}$ through $\phi^* = \partial\mathscr{C}(x_0, x_s, T)/\partial x_0$.

3.2. Cumulant expansion and volatility smile

In the case where the market fluctuations are moderately non-Gaussian, one might expect that a *cumulant expansion* around the Black–Scholes formula leads to interesting results. This cumulant expansion has been worked out in general in Ref. [9]. If one only retains the leading order correction which is (for symmetric fluctuations) proportional to the kurtosis, one finds that the price of options $\mathscr{C}(x_0, x_s, T)$ can be written as a Black–Scholes formula, but with a modified value of the volatility σ, which becomes price and maturity dependent [23]:

$$\sigma_{\mathrm{imp}}(x_s, T) = \sigma \left[1 + \frac{\kappa_T}{24}\left(\frac{(x_s - x_0)^2}{\sigma^2 T} - 1\right)\right]. \tag{17}$$

The volatility σ_{imp} is called the implied volatility by the market operators, who use the standard Black–Scholes formula to price options, but with a value of the volatility which they estimate intuitively, and which turns out to depend on the exercised price in a roughly parabolic manner, as indeed suggested by Eq. (17).

This is the so-called 'volatility smile'. Eq. (17) furthermore shows that the curvature of the smile is directly related to the kurtosis of the underlying statistical process on the scale of the maturity $T = N\tau^*$. We have tested this prediction by directly comparing the 'implied kurtosis', obtained by extracting from real option prices (on the BUND market) the volatility σ (which turns out to be highly correlated with a short time filter of the historical volatility), and the curvature of the implied volatility smile, to the historical value of the kurtosis κ_N discussed above. The result is plotted in Fig. 3, with no further adjustable parameter. The remarkable agreement between the implied and historical kurtosis, and the fact that they evolve similarly with maturity, shows that the

[4] For the general case, see Refs. [9,32–35].

market as a whole is able to correct (by trial and errors) the inadequacies of the Black–Scholes formula, and to encode in a satisfactory way both the fact that the distribution has a positive kurtosis, and that this kurtosis decays in an anomalous fashion due to volatility persistence effects. However, the real risks associated with option trading are, at present, not satisfactorily estimated. In particular, most risk-control softwares dealing with option books are based on a Gaussian description of the fluctuations.

4. Conclusion

Research on financial markets can focus on rather different aspects. We have discussed here three interrelated themes: the statistical nature of the market fluctuations, its use for option pricing and the need for a microscopic 'explicative' model which accounts for empirical observations. We have emphasised the fact that a good model of these fluctuations was crucial to estimate and control large risks. This requires in particular an adapted theory for option pricing, which goes beyond the traditional Black–Scholes dogma. We have shown that subtle statistical effects, such as the persistence of volatility fluctuations, is rather well reflected in option prices. This shows that the market as a whole behaves as an adaptive system, able to correct (through trial and error) the theoretical inadequacies of the Black–Scholes formula.

The theoretical understanding of the *tails* of the distributions is furthermore of fundamental importance, because by definition the empirical observation of rare events leads does not allow one to determine the extreme tails of the distribution with great accuracy. In this respect, an interesting aspect of the Langevin equation discussed in [37] is that crashes events appear as fundamentally distinct from 'normal' events, and their probability of occurrence is thus not expected to lie on the extrapolation of the (non-Gaussian) distribution constructed from these 'normal' events. This is actually what is observed: although the empirical tail of the distribution of the S& P 500's daily increments is pretty well fitted by an exponential down to probabilities of 10^{-3}, events which have a probability of 10^{-4} (i.e., crashes which occur once in every 40 years) have an amplitude which is much larger than expected. In this respect, a power-law decay with a larger high exponent seems to be favoured [12,13], since in this case even the biggest crashes can be satisfactorily accounted for. What is missing is a satisfactory 'microscopic' theory of markets to account for these observations (for a review of different trials, including herding, panic effects, self-adaptive strategies, etc., see Ref. [38]).

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