Turbulence and finance?

SIR: Analogies between the price dynamics in the foreign exchange market and 3-dimensional fully developed turbulence were recently presented [1]. Independently, we have carried out a parallel study comparing the parallel of the dynamical properties of the S&P 500 index and of the time evolution of a 3-dimensional fully turbulent fluid, but our study arrives at rather different conclusions. Specifically, we find while intermittency – i.e. abrupt changes of activity in the time evolution of the variance of price changes and of the mean energy dissipation – and non-Gaussian behavior (for short times) in the probability distribution of price and velocity changes characterize both systems, the stochastic nature of the two processes is *quantitatively* quite different. Among the differences we find are:

(i) *Price* changes of the S&P 500 are substantially uncorrelated (we detect only a slight enhanced diffusion), but *velocity* changes in 3-dimensional turbulence are anti-correlated. Specifically, we find that the measured autocorrelation function of the price changes is a fast-decaying monotonic function with a characteristic time of few minutes and the spectral density of the index is a power-law $S(f) \propto f^{-2}$ for more than 4 orders of magnitude, but in 3-dimensional turbulence an inertial range where $S(f) \propto f^{-5/3}$ exists (Fig.1).

(ii) The maximum of the probability distribution $P(Z=0)$ of price changes $Z_{\Delta t}(t)$ after a time Δt shows clear Lévy (non-Gaussian) scaling (for $1 \leq \Delta t \leq 1000$ minutes) as a function of Δt [2], but the corresponding turbulence quantity *does not* show scaling (Fig. 2).

One interesting result presented in [1] is that the distribution of price changes in the foreign exchange market is changing shape as a function of the time delay Δt. The distribution evolves from a leptokurtic (i.e. with tails fatter than in a Gaussian distribution) to a Gaussian shape. Although a similar scenario is observed in fully developed turbulence, the deep difference observed in the correlation (of price and velocity changes) and scaling (of the central part of the respective distributions) properties of the two processes lead us to conclude that indeed, at the moment, the simplest model describing the major features of the dynamics of prices in speculative markets is the *truncated Lévy flight* (TLF) introduced in [2–4]. In this model, independent identically distributed price changes are characterized by a Lévy stable distribution in the central part of the distribution, but a cut-off is present after which the distribution is not power-law. This truncation implies that the process is characterized by a finite variance. A TLF is only approximately self-similar and the process eventually converges to a Gaussian distributed process due to the finiteness of the variance, as predicted by the central limit theorem. The TLF also describes quite well the dynamics of a price in foreign exchange markets [5]. Features observed in economic data that are not explained in terms of the TLF model are (i) the time dependence of the scale factor parameter γ of the TLF distribution, which shows a fluctuating behavior with bursts of activity localized in limited intervals of time, and (ii) the long-range correlations observed in the time evolution of γ and in the variance of price changes (two quantities related to the what is called "volatility" in the economics literature) [6].

Can we reconcile the known intuitive parallels between finance and turbulence with the fact that quantitatively the two phenomena are quite different? This quantitative difference might disappear for a turbulent system in an abstract space of non-integral dimensionality. Theoretical studies [7,8] (we thank A. Vulpiani for suggesting these references) show that for $d \approx 2.05$ there exists (under the Taylor hypothesis) an uncorrelated turbulent behavior characterized by a spectral density $S(f) \propto f^{-2}$, just as for the S&P 500. Further study is required to test if a 2.05-dimensional turbulence could be really consistent with the stochastic properties of market data and with the main assumption of mathematical finance that is that no arbitrage is possible in an efficient market [9].

REFERENCES

[1] Ghashghaie, S. et al, *Nature* **381**, 767–769 (1996).

[2] Mantegna, R. N. and Stanley, H. E. *Nature* **376**, 46–49 (1995).

[3] Mantegna, R. N. and Stanley, H. E. *Phys. Rev. Lett.* **73**, 2946–2949 (1994).

[4] Mantegna, R. N. and Stanley, H. E "Ultra-Slow Convergence to a Gaussian: The Truncated Lévy Flight," in *Lévy Flights and Related Topics in Physics* [Proc. 1994 Int'l Conf. on Lévy Flights], edited by M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch (Springer, Berlin, 1995), pp. 300–312.

[5] Arneodo, A. et al, Submitted to *Nature*, cond-mat/9607120

[6] Mantegna, R. N. and Stanley, H. E *Scaling Approach to Finance* (to be published).

[7] Frisch, U., Lesieur, M. & Sulem, M. *Phys. Rev. Lett.* **37**, 895–897 (1994).

[8] Fournier, J.-D. & Frisch, U. *Phys. Rev.* A **17**, 747–762 (1978).

[9] Samuelson, P. A., *Industrial Management Review* Vol. 6, No. 2 , 41-49 (1965); reproduced as Chapter 198 in Samuelson, *Collected Scientific Papers, Volume III*, Cambridge, M.I.T. Press, 1972.

FIGURES

FIG. 1. (a) Standard deviation $\sigma_Z(\Delta t)$ of the probability distribution $P(Z)$ characterizing the price changes $Z_{\Delta t}(t)$ plotted double logarithmically as a function of Δt for the S&P 500 time series. After a time interval of superdiffusive behavior ($0 < \Delta t \leq 15$ minutes), a diffusive behavior close to the one expected for a random process with independent identically-distributed increments is observed; the measured diffusion exponent 0.53 is very close to the theoretical (*uncorrelated*) value 1/2. (b) Standard deviation $\sigma_U(\Delta t)$ of the probability distribution $P(U)$ characterizing the velocity changes $U_{\Delta t}(t)$ plotted double logarithmically as a function of Δt for the velocity difference time series in turbulence (experimental data were kindly provided to us by Prof. K.R. Sreenivasan). Data recorded in the atmosphere at a Taylor microscale Reynolds number R_λ of the order of 1500. After a time interval of superdiffusive behavior ($0 < \Delta t \leq 10$), a diffusive behavior close to the one expected for a fluid in the inertial range is observed (the measured diffusion exponent 0.33 is close to the theoretical (*anti-correlated*) value 1/3). (c) Spectral density of the S&P 500 time series for the time period 1984–1987 representative of the 6-year time period 1984–1989 (an investigation performed for the time period 1986–1989 gives a curve overlapping with the shown figure). The $1/f^2$ power-law behavior expected for a random process with independent increments is observed over a frequency interval of more than 4 orders of magnitude. (d) Spectral density of the velocity time series of a 3-dimensional fully developed turbulent fluid. The $1/f^{5/3}$ inertial range (low frequency) and the dissipative range (high frequency) are clearly observed.

FIG. 2. (a) "Probability of return to the origin" $P(Z=0)$ for the S&P 500 Index ($\circ$) and $P_g(Z=0) = 1/\sqrt{2\pi}\sigma(\Delta t)$ (filled squares) as functions of the time sampling interval Δt. $P_g(Z=0)$ is the probability of return to the origin expected for a Gaussian stochastic process determined by measuring the standard deviation $\sigma(\Delta t)$ of the experimental data. The two measured quantities differ in the full interval implying that the profile of the PDF must be *non-Gaussian*. A power-law behavior is observed for the entire time interval spanning three orders of magnitude. The slope of the best linear fit is -0.71 ± 0.025. The difference between the two quantities is decreasing when Δt increases, implying a convergence to a Gaussian process for high values of Δt. (b) Probability of return to the origin $P(0)$ ($\circ$) and $P_g(0)$ (filled squares) (defined as in (a)) as functions of the time sampling interval Δt for the velocity of the fully turbulent fluid. Again, the two measured quantities differ in the full interval, implying that the profile of the PDF must be non-Gaussian. However in this case, a single scaling power-law behavior does not exist for the entire time interval spanning three orders of magnitude. The slope of the best linear fit (which is of quite poor quality) is -0.59 ± 0.11.