Volatility distribution in the S&P500 Stock Index

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Abstract

We study the volatility of the S&P500 stock index from 1984 to 1996 and find that the volatility distribution can be very well described by a log-normal function. Further, using detrended fluctuation analysis we show that the volatility is power-law correlated with Hurst exponent $\alpha \cong 0.9$.

The volatility is a measure of the mean fluctuation of a market price over a certain time interval T. The volatility is of practical importance since it quantifies the risk related to assets [1]. Unlike price changes that are correlated only on very short time scales [2] (a few minutes), the absolute values of price changes (which are closely related to the volatility) show correlations on time scales up to many years [3–5].

Here we study in detail the volatility of the S&P 500 index of the New York stock exchange, which represents the stocks of the 500 largest U.S. companies. Our study is based on a data set over 13 years from January 1984 to December 1996 reported at least every minute (these data extend by 7 years the data set previously analyzed in [6]).

We calculate the logarithmic increments

$$G(t) \equiv \log_e Z(t + \Delta t) - \log_e Z(t) , \tag{1}$$

where $Z(t)$ denotes the index at time t and Δt is the time lag; $G(t)$ is the relative price change $\Delta Z/Z$ in the limit $\Delta t \to 0$. Here we set $\Delta t = 30\,\mathrm{min}$, well above the correlation time of the price increments; we obtain similar results for other choices of Δt (larger than the correlation time).

Over the day, the market activity shows a strong "U-shape" dependence with high activity in the morning and in the afternoon and much lower activity over noon. To remove artificial correlations resulting from this intra-day pattern of the volatility [7–10], we analyze the normalized function

$$g(t) \equiv G(t)/A(t) \,, \tag{2}$$

where $A(t)$ is the mean value of $|G(t)|$ at the same time of the day averaged over all days of the data set.

We obtain the volatility at a given time by averaging $|g(t)|$ over a time window $T = n \cdot \Delta t$ with some integer n,

$$v_T(t) \equiv \frac{1}{n} \sum_{t'=t}^{t+n-1} |g(t')| \,. \tag{3}$$

Figure 1 shows (a) the S&P500 index and (b) the signal $v_T(t)$ for a long averaging window $T = 8190\,\mathrm{min}$ (about 1 month). The volatility fluctuates strongly showing a marked maximum for the '87 crash. Generally periods of high volatility are not independent but tend to "cluster". This clustering is especially marked around the '87 crash, which is accompanied by precursors (possibly related to the oscillatory patterns postulated in [11]). Clustering occurs also in other periods (e.g. in the second half of '90), while there are extended periods where the volatility remains at a rather low level (e.g. the years of '85 and '93).

Fig. 2a shows the scaled probability distribution $P(v_T)$ for several values of T. The data for different averaging windows collapse to one curve. Remarkably, the scaling form is log-normal, not Gaussian. In the limit of very long averaging times, one expects that $P(v_T)$ becomes Gaussian, since the central limit theorem holds also for correlated series [12], with a slower convergence than for non-correlated processes [13,14]. For the times considered here, however, a log-normal fits the data better than a Gaussian, as is evident in Fig. 2b which compares the best log-normal fit with the best Gaussian fit for data averaged over a 300 min window.

The correlations found in Fig.1b can be accurately quantified by detrended fluctuation analysis [15]. The analysis reveals power-law behavior independent of the T value chosen (Fig. 3) with an exponent $\alpha \cong 0.9$ in agreement with the value found for the absolute price increments [5].

To account for the time dependence of the volatility and its correlations, ARCH [16], GARCH [17] models and related approaches [18] have been developed, which assume that the volatility depends on time and on the past evolution of the index. It may be worthwhile to test models also for the volatility distribution $P(v_T)$.

In summary, we have found that the probability distribution of the S&P500 volatility can be well described by a log-normal function and that the volatility shows power-law correlations with Hurst exponent $\alpha \cong 0.9$. The log-normal shape of the distribution is consistent with a multiplicative process [19] for the volatility [20]. However, a multiplicative behavior would be surprising, because efficient market theories [2] assume that the price changes, $G(t)$, are caused by incoming new informations about an asset. Since such information-induced price changes are additive in $G(t)$, they should not give rise to multiplicative behavior of the volatility.

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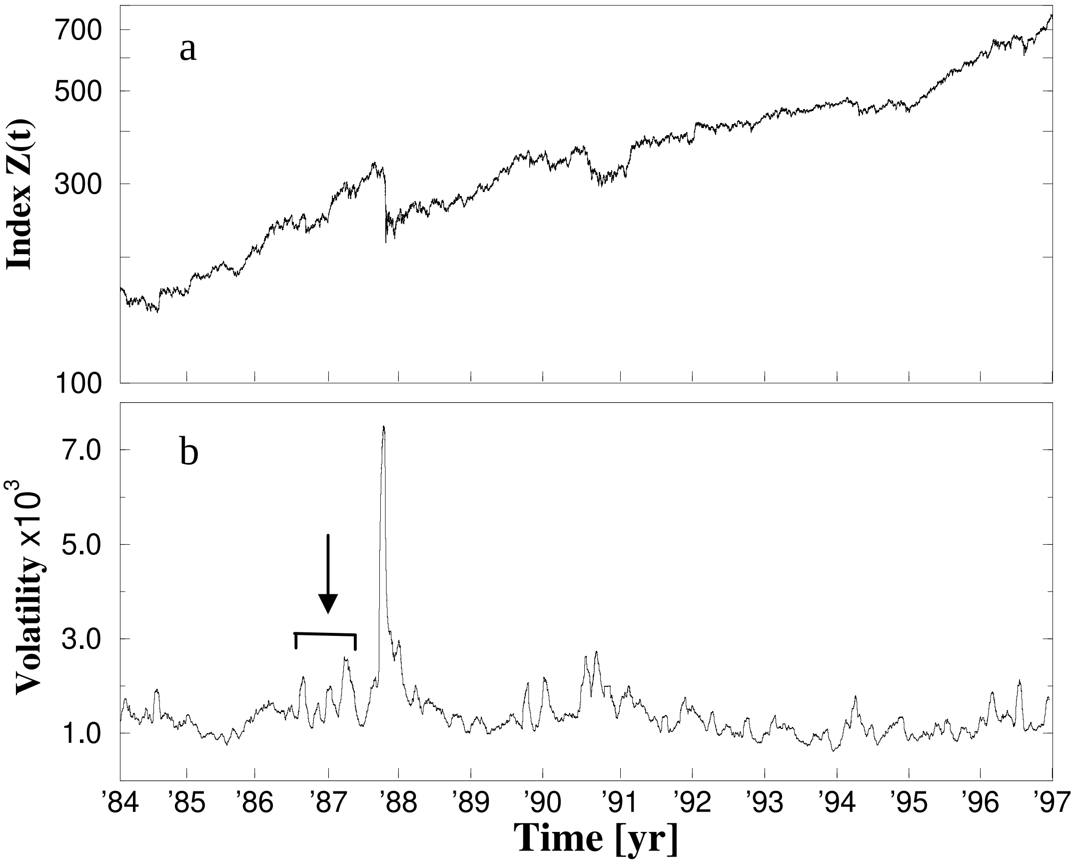

Fig. 1. (a) Raw data analyzed: The S&P 500 index Z(t) for the 13-year period 1 Jan 1984 - 31 Dec 1996 at interval of 1 min. Note the large fluctuations, such as that on 19 Oct 1987 ("black Monday"). (b) Volatility $v_T(t)$ with T=1mon (8190min) and time lag 30min. The precursors of the '87 crash are indicated by arrows.

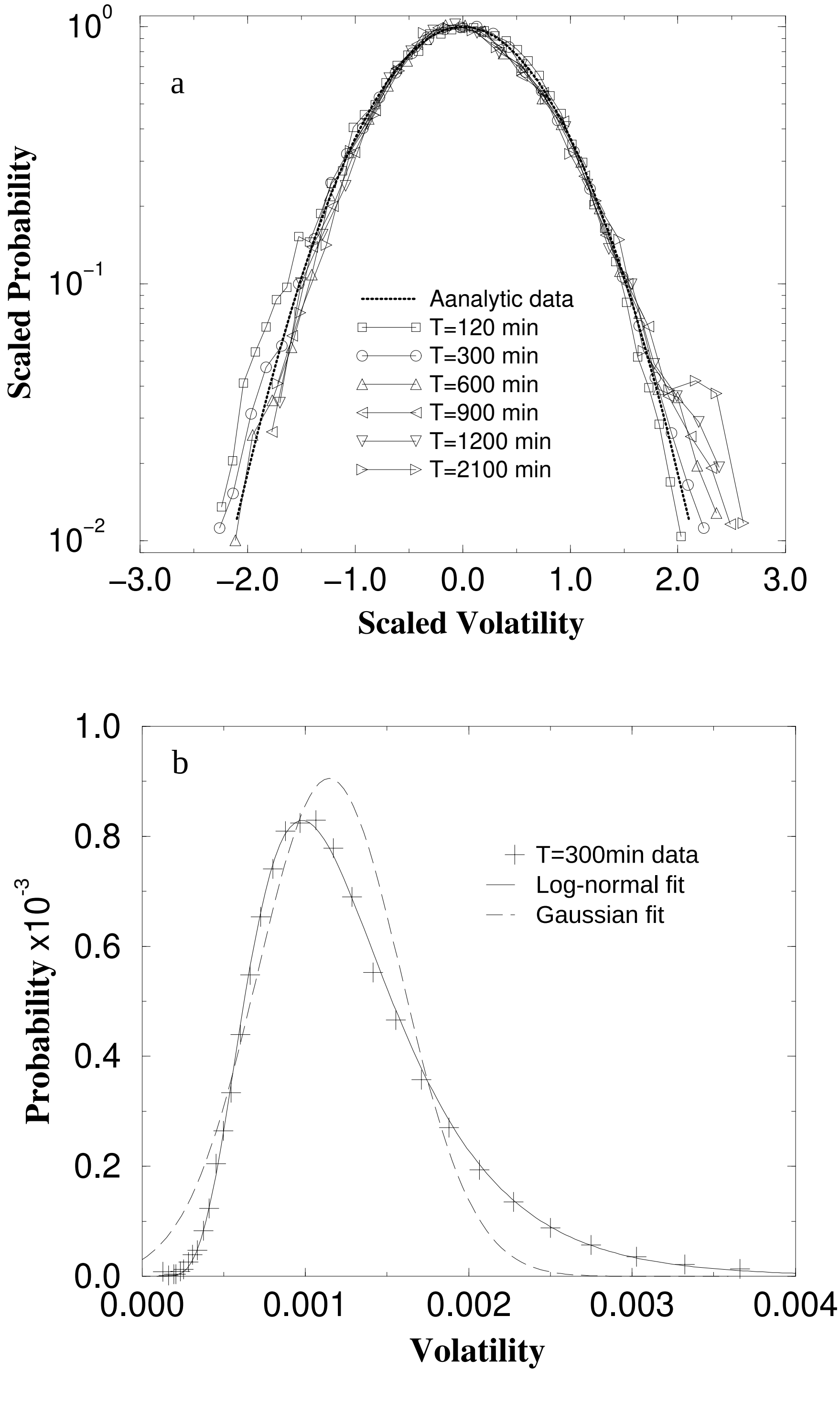

Fig. 2. (a) The volatility distribution for different window sizes T in scaled form, $\sqrt{\mu}\exp(a+\mu/4)P(v_T)$ as a function of $(\ln(v_T)-a)/\sqrt{\pi\mu}$, where a and μ are the mean and the width on a logarithmic scale. By the scaling, all curves collapse to the log-normal form with $a=0$ and $\mu=-1$, $\exp(-(\ln x)^2)$ (dotted line). (b) Comparison of the best log-normal and Gaussian fits for the 300min data.

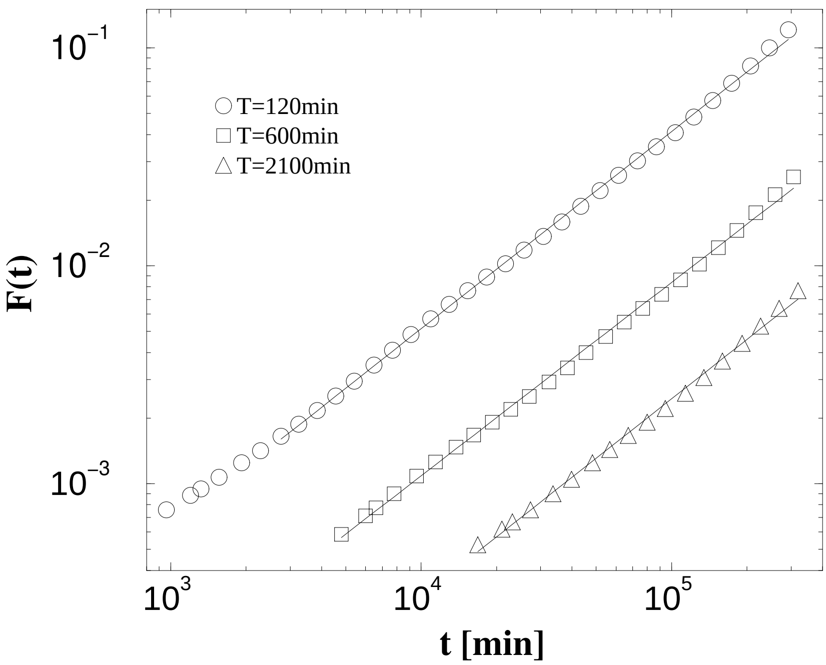

Fig. 3. The fluctuation $F(t)$ of volatility $v_T(t)$ with T=120 min, 600 min, and 2100 min, calculated using detrended fluctuation analysis (DFA) [15]. To implement the DFA method, we integrate $v_T(t)$ once; then we determine the fluctuations $F(t)$ of the integrated signal around the best linear fit in a time window of size t. The lines are the best power-law fits with exponents $\alpha =$0.91, 0.89, and 0.91.