Continuum states using

the hypersherical harmonic adiabatic method

Paolo Barletta and Alejandro Kievsky

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1 – Overview

- the variational method
- introduction to the three-body system
- the hyperspherical adiabatic basis
- the adiabatic equations
- results and conclusions

2 – Variational method

Rayleigh-Ritz variational principle

N basis functions $\{\phi_i\}$, $\psi_j = \sum_i C_i^{(j)} \phi_i$

$\sum_i (\mathcal{H}_{ik} - \varepsilon_j \mathcal{N}_{ik}) C_i^{(j)} = 0 \Rightarrow$ best $C_i^{(j)}$

$(\mathcal{H}_{ik} = \langle \phi_i | \mathcal{H} | \phi_k \rangle, \mathcal{N}_{ik} = \langle \phi_i | \mathcal{N} | \phi_k \rangle)$

$\Rightarrow \varepsilon_j \geq E_j$

Kohn variational principle

$\psi = \psi_c + \psi_a = \sum_i C_i \phi_i + (g + \mathcal{L} f)$

$[\mathcal{L}, E] = \mathcal{L} - 2\langle \psi | \mathcal{H} - E | \psi \rangle$

3 – Three-body: Introduction

☞ **Jacobi coordinates** $\{x_i, y_i, \mu_i\}$ **:**

$$\vec{x}_i = \frac{\vec{r}_j - \vec{r}_k}{\sqrt{2}};\ \vec{y}_i = \sqrt{\frac{3}{2}}\left(\frac{\vec{r}_j + \vec{r}_k}{2} - \vec{r}_i\right);\ \mu_i = \hat{x}_i \cdot \hat{y}_i.$$

☞ **Hyperspherical coordinates** $\{\rho, \theta_i, \mu_i\}$ **:**

$$\rho^2 = \sqrt{\frac{1}{3}}(r^2 + s^2 + t^2) = x_i^2 + y_i^2$$

$$\theta_i = \tan^{-1}(y_i/x_i)$$

☞ **Jacobian**

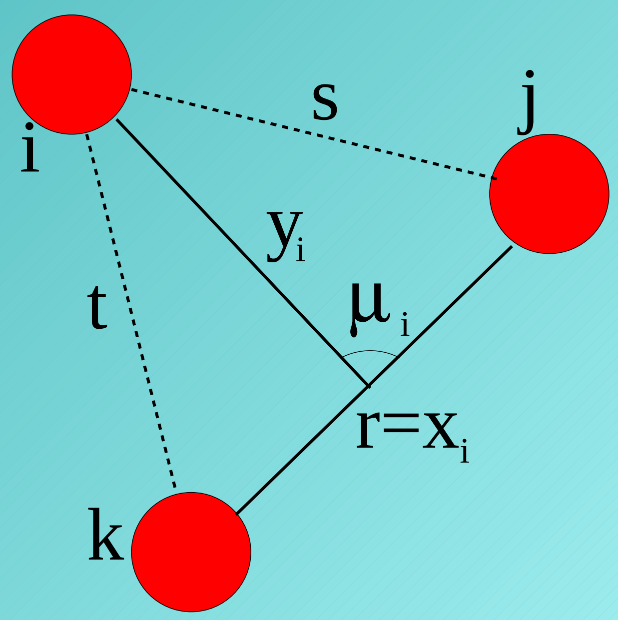
$$\rho^5 \cos^2\theta_i \sin^2\theta_i d\theta_i d\mu_i d\rho$$

☞ **Kinetic energy operator :**

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{d\rho^2} + \frac{5}{\rho}\frac{d}{d\rho}\right) + \frac{L}{\rho^2}$$

☞ **Potential operator :**

$$V = V(s) + V(t) + V(r)$$



4 – Hyperspherical approach:

$$\mathcal{H} = T_\rho + \frac{1}{\rho^2} L(\theta_i, \mu_i, ...) + V(s, r, t)$$

$$L(\Omega) B_K(\Omega) = \lambda_K B_K(\Omega)$$

$$\Psi = \sum_K u_K(\rho) B_K(\Omega)$$

$$\sum_{K'} [(T_\rho + \lambda_K/\rho^2 - E)\delta_{K',K} + V_{K',K}(\rho)] u_K(\rho) = 0 \ \ (K = 1, 2, \ldots)$$

$$\Psi = \sum_{Kn} A_{nK} L_n(\rho) B_K(\Omega)$$

$$N = N_B N_L$$

eigenvalue problem

5 – Hyperspherical adiabatic approach I:

$$\mathcal{H}_\Omega = \frac{1}{\rho^2} L(\Omega) + V(s, r, t)$$

$$\mathcal{H}_\Omega \phi_\nu(\rho; \Omega) = U_\nu(\rho) \phi_\nu(\rho; \Omega)$$

$$\{\phi_\nu(\rho; \Omega), U_\nu(\rho)\}, \quad \rho = \{\rho_i\}$$

$$\phi_\nu(\rho, \Omega) = \sum_K A_K^\nu(\rho) B_K(\Omega)$$

$$\sum_{K'} [(\lambda_K/\rho^2 - U_\nu)\delta_{K',K} + V_{K',K}(\rho)] A_K^\nu(\rho) = 0 \ \ (K = 1, 2, \ldots)$$

6 – Hyperspherical adiabatic approach II:

$$\mathcal{H}_\Omega \phi_\nu(\rho;\Omega) = U_\nu(\rho)\phi_\nu(\rho;\Omega)$$

$$\Psi(\rho,\Omega) = \sum_\nu u_\nu(\rho)\ \phi_\nu(\rho;\Omega)$$

$$\sum_\nu [(T_\rho + U_\lambda(\rho) - E)\delta_{\lambda,\nu} + B_{\lambda,\nu}(\rho) + C_{\lambda,\nu}\tfrac{d}{d\rho}]u_\nu(\rho) = 0 \ \ (\lambda = 1,2,\ldots)$$

$$B_{\lambda,\nu}(\rho) = \langle\phi_\lambda|T_\rho|\phi_\nu\rangle, \ \ C_{\lambda,\nu}(\rho) = 2\langle\phi_\lambda|\tfrac{d}{d\rho}|\phi_\nu\rangle.$$

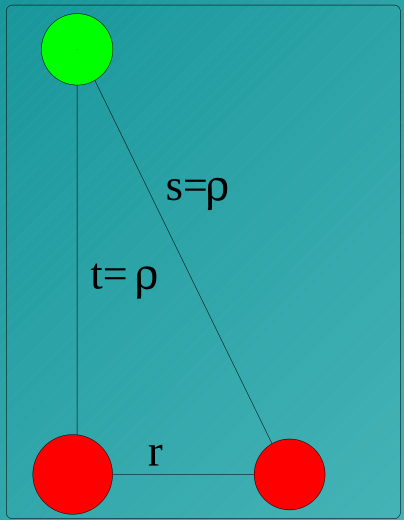
$$\{T_\rho + U_1(\rho) + B_{1,1}(\rho) - E\}u_1(\rho) = 0$$

size of the problem : N=N$_L$ N$_{eq}$

if N$_{eq}$=N$_B$ adiab. appr. eqv HH expansion

in general $N_{eq} \ll N_B$

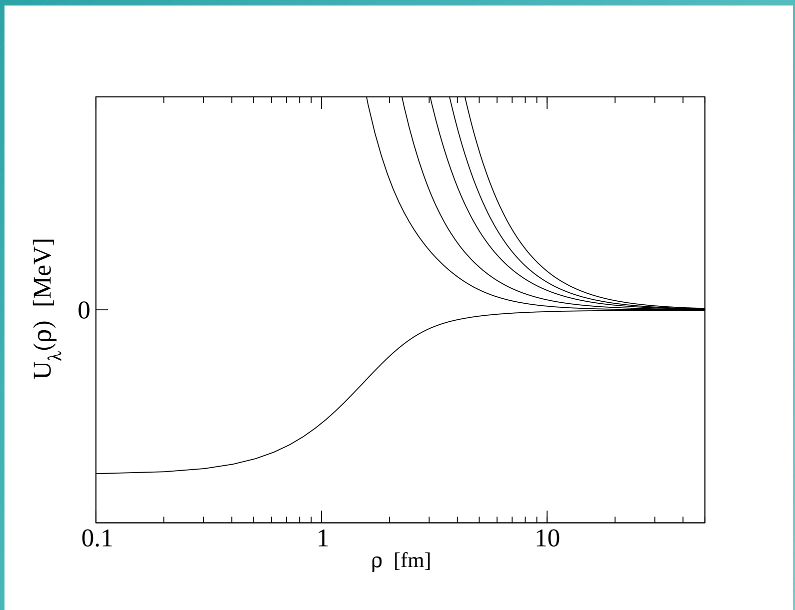
7 – Adiabatic Potentials

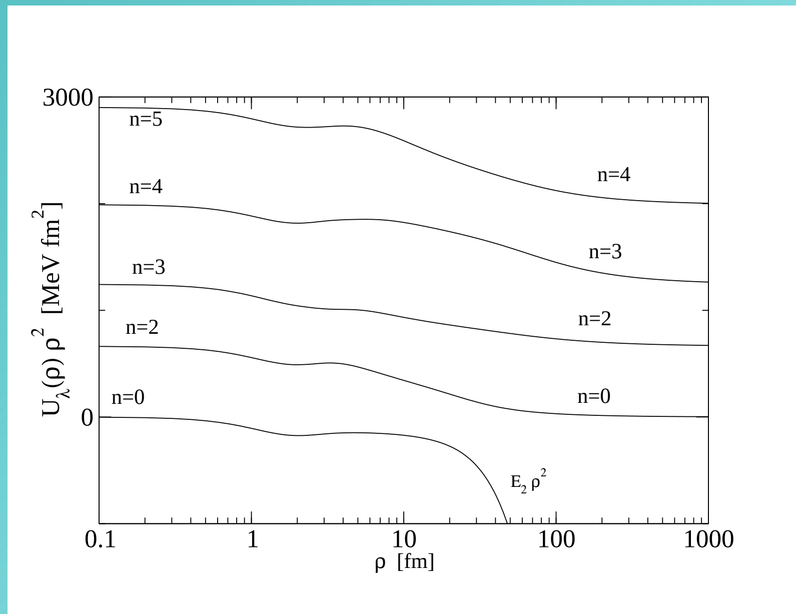


$\rho \to 0, V \to$ const. $\{U_\lambda\} = \{U_n^{(0)}\}$

$\rho \to \infty$ $\{U_\lambda\} = \{U_n^{(0)}, E_2\}.$

recalling that $\rho^2 \propto s^2 + t^2 + r^2$





8 – Asymptotic behaviour of $\{U(\rho)\}$:

$$\mathcal{H}_\Omega \phi_\nu(\rho;\Omega) = U_\nu(\rho)\phi_\nu(\rho;\Omega)$$

$$\phi \rightarrow g/(\cos\theta_i \sin\theta_i)$$

$$\rho \gg r_0 \Rightarrow V(s) = V(t) \approx 0, \;\; \ell = 0$$

$$\left\{-\frac{1}{2\rho^2}\frac{d^2}{d\theta^2} + V(r)\right\} g = 0$$

$$\sqrt{2}\rho\,(\pi/2 - \theta_i) \rightarrow w$$

$$\left\{-\frac{d^2}{dw^2} + V'(w)\right\} g = 0$$

$$U_0(\rho) \approx E_{2B} + c/\rho^2 + \ldots$$

$$U_n(\rho) \approx \lambda_K/\rho^2 + d/\rho^3 + \ldots$$

similar expansions for $B_{\nu,\lambda}, C_{\nu,\lambda}$

9 – Strategy

- the hyperspherical hamiltonian ($\mathcal{H}_\Omega$) is solved on a grid $\{\rho_i\}$
- the Ω basis set can be optimized for different ρ_i
- $\rho \geq \rho_M$: two-body problem
- the appropriate variational principle is applied
- convergence on adiabatic channels is seeked

Lowest adiabatic channel accounts for at least 90% of the desired observable!

10 – Results: bound and scattering states with CP

$$V(r) = A \exp\left[-(r/d)^2\right] \quad A = 51.5 \text{ MeV} \quad d = 1.6 \text{ fm}$$

$$\ell = 0,\ E = -0.39774 \text{ MeV},\ a_s = 11.36 \text{ fm}$$

bound states

N	E_1 (MeV)	E_2 (MeV)
1	-9.7559	-0.4781
2	-9.7767	-0.4810
3	-9.7794	-0.4814
4	-9.7795	-0.4815
5	-9.7796	-0.4815

scatt states ($\mathcal{L} = \tan\theta$)

N	-0.30 (MeV)	-0.25 (MeV)	-0.15 (MeV)	-0.05 (MeV)
4	-76.3256	87.4769	78.2328	68.8296
8	-76.2439	87.3885	78.3206	68.9301
12	-76.2348	87.3795	78.3273	68.9387
16	-76.2329	87.3779	78.3279	68.9400
20	-76.2322	87.3773	78.3280	68.9403

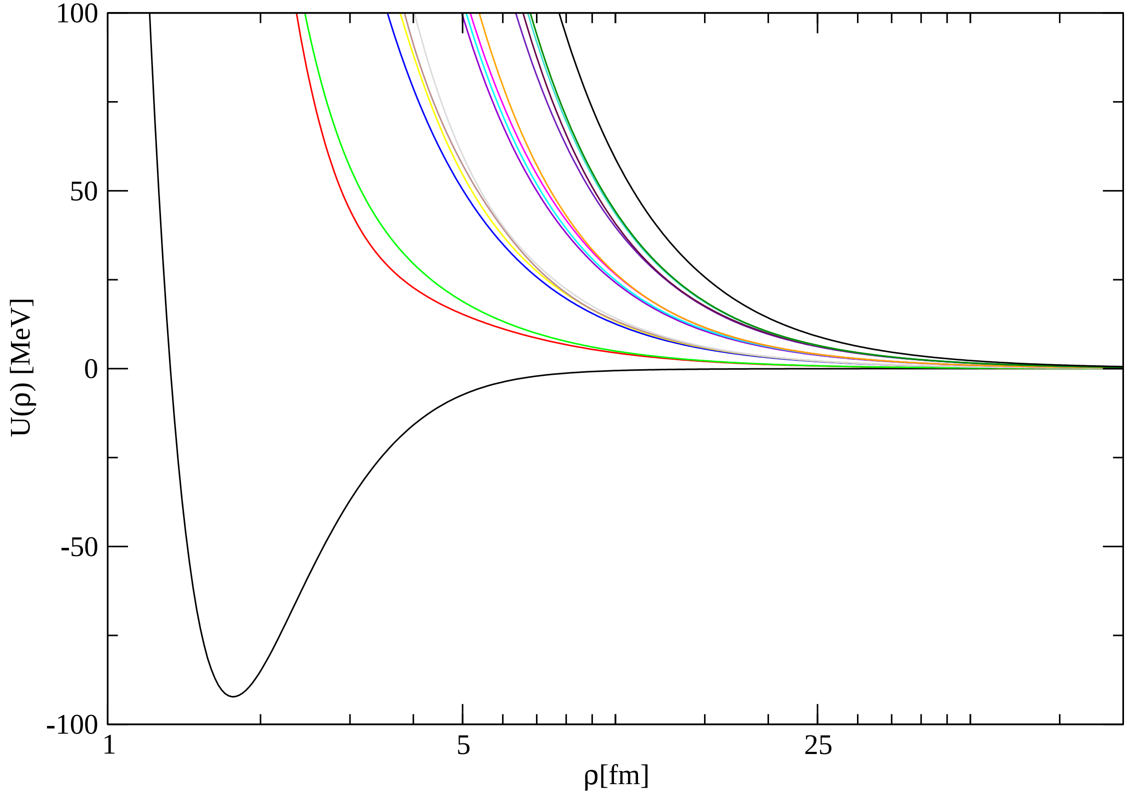
Continuum states ...

11 – Results: bound states with RP

nnp, potential AV14

N_c^1	E (MeV)	
	ad	HH
3	-3.563	-3.564
4	-4.130	-4.131
5	-4.138	-4.139
6	-4.174	-4.176
7	-4.342	-4.346
8	-4.424	-4.428

[1] $K_{max} = 10$



12 – Conclusions

- In this work, we have applied the hyperspherical adiabatic (HA) method to the study of bound and scattering states of a three-nucleon system.
- The HA method operates as a two-step procedure.
- First, a set of adiabatic functions and potentials are obtained on a grid $\{\rho_i\}$,
- then a set of coupled differential eqs is solved (which is problem-independent)
- the size of the problem is reduced considerably ($N_{eq} \ll N_B$)
- however, price to pay are numerical complications.
- Future work will address a full convergence for the realistic case
- and scattering at higher energies.