# Universality in few-body systems: from few atoms to few nucleos

A. Kievsky

INFN, Sezione di Pisa (Italy)

XIV Convegno su Problemi di Fisica Nucleare Teorica
Cortona, Ottobre 2013

## Collaborators

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# Universality in the two-body system

## Shallow states

- Low energy scattering $E = \hbar^2 k^2/m << \hbar^2/mr_0^2$

$$k \cot\delta = -\frac{1}{a} + \frac{1}{2} r_0 k^2$$

- $a >> r_0 \rightarrow$ shallow dimer $E_D \approx \hbar^2/ma^2$
- $r_0 \rightarrow 0$, $k \cot\delta = -1/a$ and $E_D = \hbar^2/ma^2$

## Continuous Scale Invariance

- $\Psi_D \rightarrow \frac{e^{-r/a}}{r}$ $\qquad < r^2 >= a^2/2 \qquad \sigma = 2\pi \frac{4a^2}{1+a^2k^2}$
- $a \rightarrow \lambda a$

  $E \rightarrow \lambda^{-2} E$

  $\sigma \rightarrow \lambda^2 \sigma$

  $< r^2 > \rightarrow \lambda^2 < r^2 >$

In $N = 2$ we have CSI $\rightarrow$ $N > 2$ Discrete Scale Invariance (DSI)

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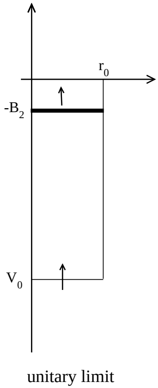
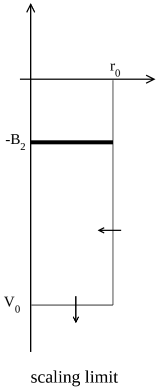
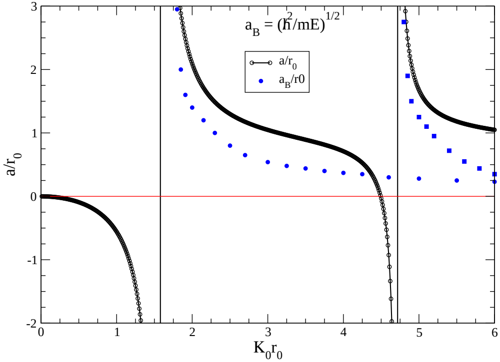
# Efimov Effect (1970) and Thomas Effect (1935)




$\frac{r_0}{a} << 1$ $\quad B_2 \approx \hbar^2/ma^2$ $\quad B_2 = \hbar^2/ma^2$

# Natural and unnatural states (shallow states)




Most of the time $E \approx -\hbar^2/mr_0^2$ $\rightarrow$ natural state
In same cases $a >> r_0$ and $E \approx -\hbar^2/ma^2$ $\rightarrow$ fine tunning

# $N = 3$

## Thomas collapse (scaling limit)

- $r_0 \to 0$, $V_0 \to -\infty$
- $B_2 = constant \approx \hbar^2/ma^2$
- The three-body bound state $E_3 \to -\hbar^2/mr_0^2 \to -\infty$
- The three-body bound state is unbound from below
- curiosity: even if $B_2 \approx 0$, $E_3 \to -\infty$

## Efimov effect (unitary limit)

- $E_2 \to 0$, $a_0 \to \infty$
- $r_0 = constant$
- a series of states appears between $-\hbar^2/mr_0^2 \leq E_3^n \leq -\hbar^2/ma^2$
- two consecutive states have ratio: $E_3^{(n+1)}/E_3^{(n)} \to e^{-2\pi/s_0}$
- The series of states is infinite at the unitary limit
- $s_0$ is an universal number

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# The Efimov Effect

- The three body Hamiltonian:

$H = T + V(1,2) + V(2,3) + V(3,1)$

- Hyperspherical coordinates:

$\rho^2 = x^2 + y^2 \qquad \boldsymbol{x} = \boldsymbol{r}_1 - \boldsymbol{r}_2\,, \qquad \boldsymbol{y} = \boldsymbol{r}_3 - (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2\,,$

$[\boldsymbol{x},\boldsymbol{y}] \equiv [\rho,\Omega] = [\rho,\hat{x},\hat{y},\arctan(x/y)] = [\rho,\hat{x},\hat{y},\alpha]$

- The kinetic energy:

$$T = T_\rho - \frac{L^2(\Omega)}{\rho^2} = \frac{\partial^2}{\partial\rho^2} + \frac{5}{\rho}\frac{\partial}{\partial\rho} - \frac{L^2(\Omega)}{\rho^2}$$

- Adiabatic Hyperspherical expansion:

$$\Psi(\boldsymbol{x},\boldsymbol{y}) = \Psi(\rho,\Omega) = \rho^{-5/2}\sum_n w_n(\rho)\phi_n(\rho,\Omega)$$

- the adiabatic functions $\phi_n(\rho,\Omega)$ are solutions of:

$$\left(\frac{\hbar^2}{m}\frac{L^2(\Omega)}{\rho^2} + V(1,2) + V(2,3) + V(3,1)\right)\phi_n(\rho,\Omega) = U_n(\rho)\phi_n(\rho,\Omega)$$

# The Efimov Effect

- The hyperadial functions $w_n(\rho)$ are solutions of:

$$\left[\frac{\hbar^2}{m}\left(-\frac{\partial^2}{\partial\rho^2}+\frac{15}{4\rho^2}\right)+U_n(\rho)\right]w_n(\rho)+\sum_m C_{nm}w_m(\rho)=Ew_n(\rho)$$

- The equation for the lowest hyperadial functions $w_0(\rho)$ as $\rho\to\infty$ is:

$$\frac{\hbar^2}{m}\left(-\frac{\partial^2}{\partial\rho^2}-\frac{s_0^2+1/4-\rho^2/a^2}{\rho^2}\right)w_0(\rho)=Ew_0(\rho)$$

- in the unitary limit $a\to\infty$ and the solutions are

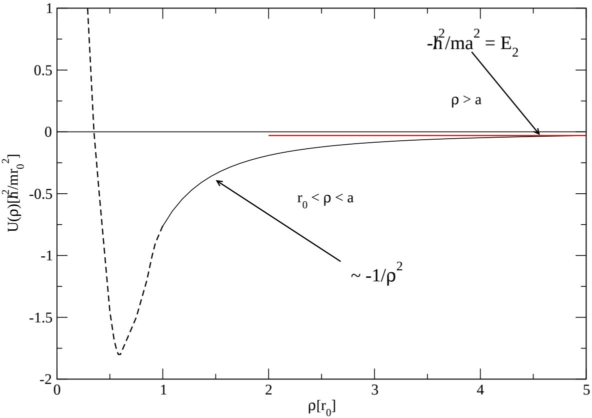
$w_0(\rho)=\rho^{1/2}K_{is_0}(\sqrt{2}\kappa\rho)$ with $\kappa^2=E/(\hbar^2/m)$

- The discrete spectrum arises from the boundary condition at $\rho\approx 0$

$$E^{(i+1)}=\mathrm{e}^{-2\pi/s_0}E^{(i)}$$

• a number of discrete states appears between $\hbar^2/mr_0^2 < E_i < \hbar^2/ma^2$

$$N \to \frac{s_0}{\pi} \ln \frac{|a|}{r_0}$$



# Three-boson spectrum for $r_0/a << 1$

## Zero-Range Theory

$$E_3^n/(\hbar^2/ma^2) = \tan^2\xi$$

$$\kappa_* a = e^{\pi(n-n_*)/s_0} e^{-\Delta(\xi)/2s_0} / \cos\xi$$

- $\kappa_*$ is the three-body parameter
- $\Delta(\xi)$ is an universal function

$$E_3^n + \frac{\hbar^2}{ma^2} = \left[e^{-2\pi(n-n_*)/s_0}\right] \left[e^{\Delta(\xi)/s_0}\right] \frac{\hbar^2\kappa_*^2}{m}$$

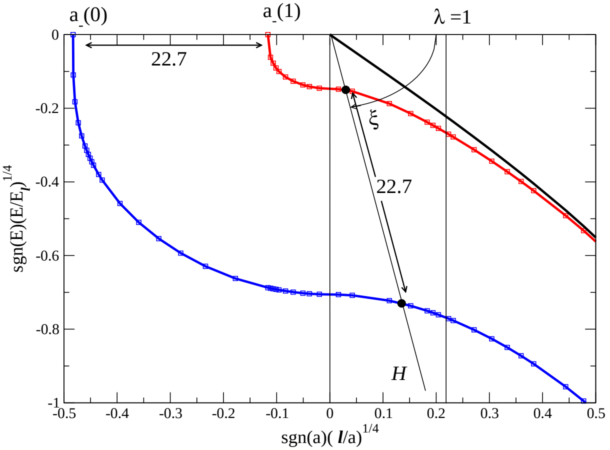
$$E_3^{n_*} = \frac{\hbar^2\kappa_*^2}{m}$$

- the ratios at $\xi = const.$ are $E_3^{(n+1)}/E_3^{(n)} = e^{-2\pi/s_0} \approx 1/22.7^2$

# Discrete Scale Invariance

Polar Coordinates: $1/a = H\cos\xi\,,\quad K = H\sin\xi$

$$E_3^n + \frac{\hbar^2}{ma^2} = e^{-2n\pi/s_0} e^{\Delta(\xi)/s_0}\frac{\hbar^2\kappa_*^2}{m} \rightarrow H = \kappa_*\, e^{-n\pi/s_0}\, e^{\Delta(\xi)/2s_0}$$



# Studying Universality

## Efimov physics with potential models

- Tunable strength
- Finite-range versus zero-range theory
- Finite-range effects
- Equivalent to results from EFT

## Discrete Scale Invariance

- Efimov effect
- Three-boson continuum
- Evolution with $N$
- Efimov physics is a well established sector of Quantum Mechanics
  Quantum Mechanics of shallow states
- Experimental activities in ultracold trapped atoms
- Experimental in nuclear physics

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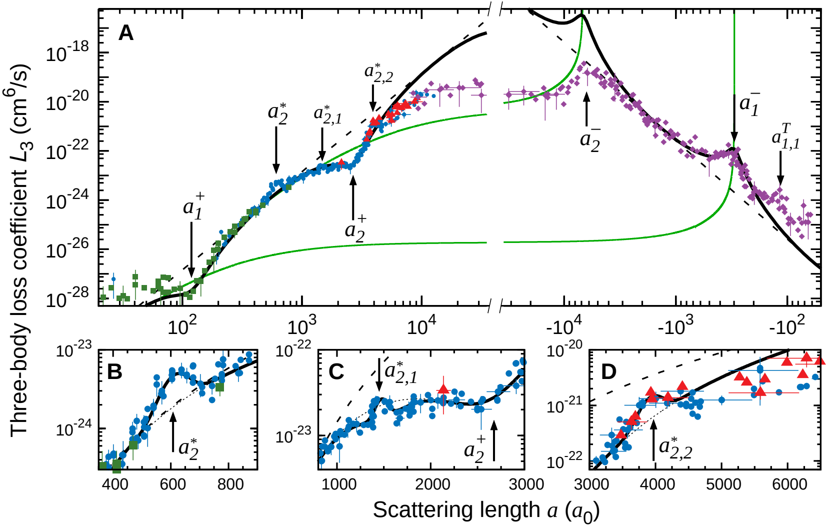
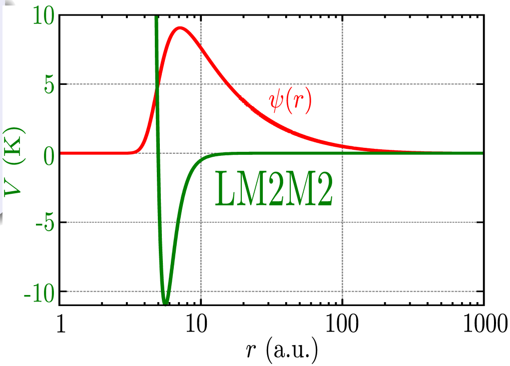
S.E. Pollack et al., Science 326 (2009) 1683



## Efimov physics with potential models

The He-He system as example:

$E(\mathrm{He}-\mathrm{He}) \approx -1.3$ mK
$a = 190$ a.u.
$r_0 = 13$ a.u.
$a >> r_0$
$E(\mathrm{He}-\mathrm{He}) \approx -\frac{\hbar^2}{m}\frac{1}{a^2}$
$\approx -1.2$ mK



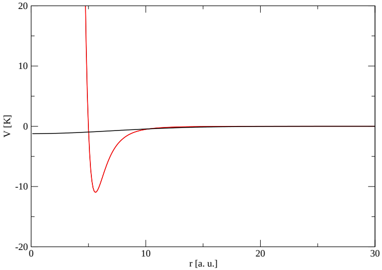
# Soft Two-Body Gaussian Potential

## Effective low-energy soft potential

- $V(r) = V_0 \, \mathrm{e}^{-r^2/R^2}$
  - Regularized contact interaction
  - Fix $V_0$ to reproduce one low-energy LM2M2 datum
  - Use the cut-off $R$ to reproduce a second datum

$V_0 = -1.2344$ K, $R = 10.0$ a.u.

| | Gaussian | LM2M2 |
|---|---|---|
| $a_0$ (a.u.) | 189.41 | 189.42 |
| $r_0$ (a.u.) | 13.81 | 13.84 |
| $E_2$ (mK) | -1.303 | -1.303 |



# Soft Hyper-Central Three-Body Potential

## Problem in the three-body sector

| | Soft-Gaussian | LM2M2 |
|---|---|---|
| $E_3^{(0)}$ (mK) | -150.4 | -126.4 |
| $E_3^{(1)}$ (mK) | -2.467 | -2.271 |

## Effective low-energy three-body-soft potential

$$W(\rho_{ijk}) = W_0 \, e^{-2\rho_{ijk}^2/\rho_0^2} \quad (\rho_{ijk}^2 \propto r_{ij}^2 + r_{jk}^2 + r_{ki}^2)$$

| potential | $E_{3b}^{(0)}$ (mK) | $E_{3b}^{(1)}$ (mK) |
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| LM2M2 | $-126.4$ | $-2.265$ |
| gaussian | $-150.4$ | $-2.467$ |
| ($W_0$ [K], $\rho_0$ [a.u.]) | | |
| $(0.422, 14)$ | $-126.4$ | $-2.299$ |

- LO Effective Field Theory

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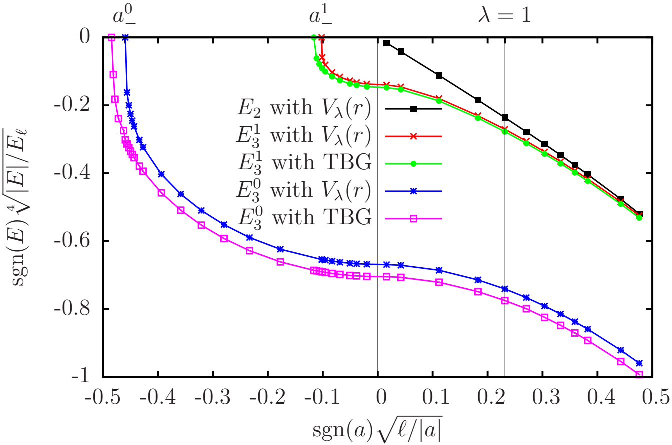
$V_\lambda(r) = \lambda V_{LM2M2}(r)$

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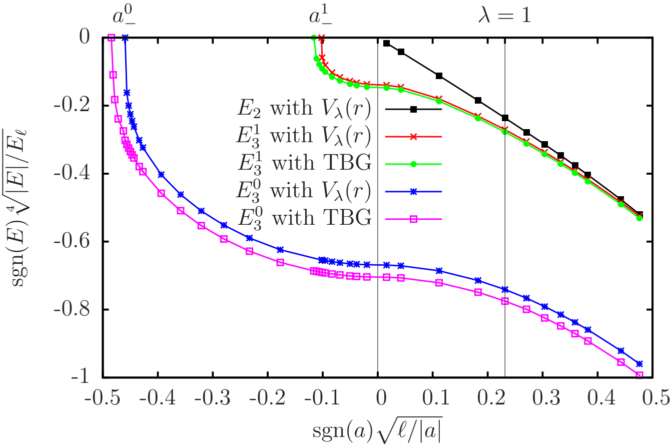
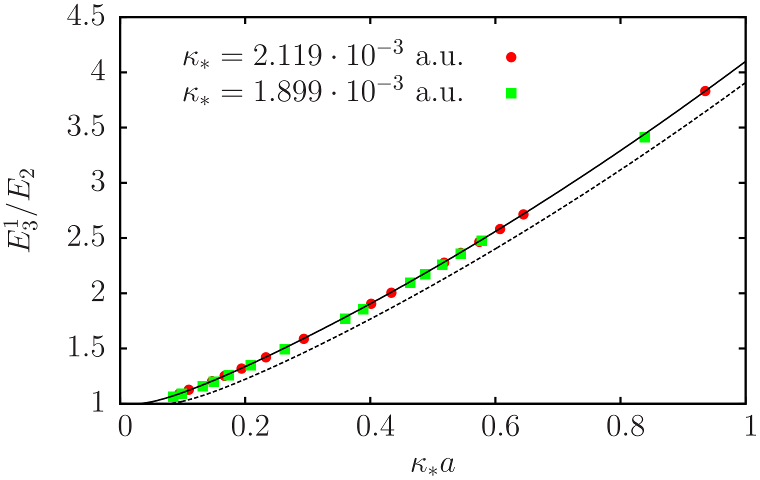



- Soft Potentials together with the Schrödinger equation can be used to investigate Efimov physics

# Potential calculations vs. Universal theory

$E_3^1/(\hbar^2/ma^2) = \tan^2\xi$
$\kappa_* a = e^{-\Delta(\xi)/2s_0} / \cos\xi$



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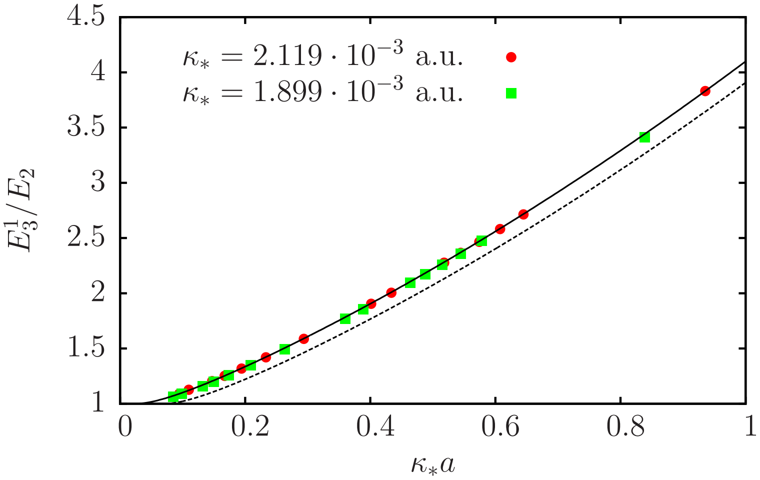
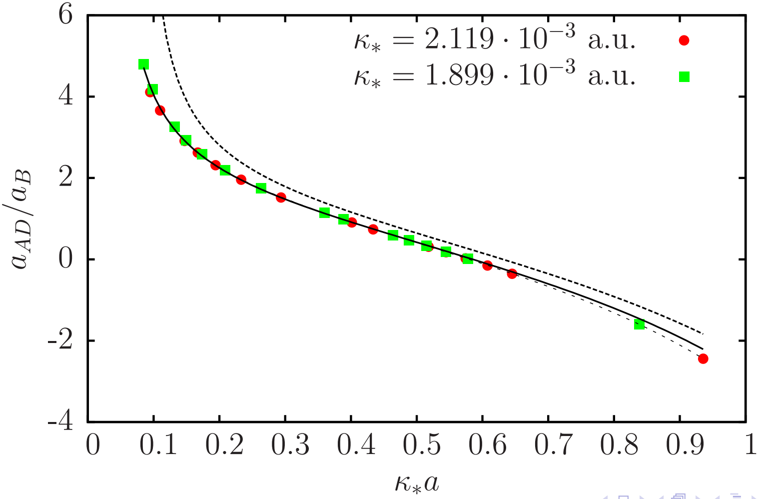


# Universality in atom-dimer scattering

$a_{AD} = a(d_1 + d_2 \tan[s_0 \ln(a\kappa_*) + d_3])$ (Efimov 1979)
with $d_1$, $d_2$, $d_3$ universal constants (Braaten and Hammer, 2006)



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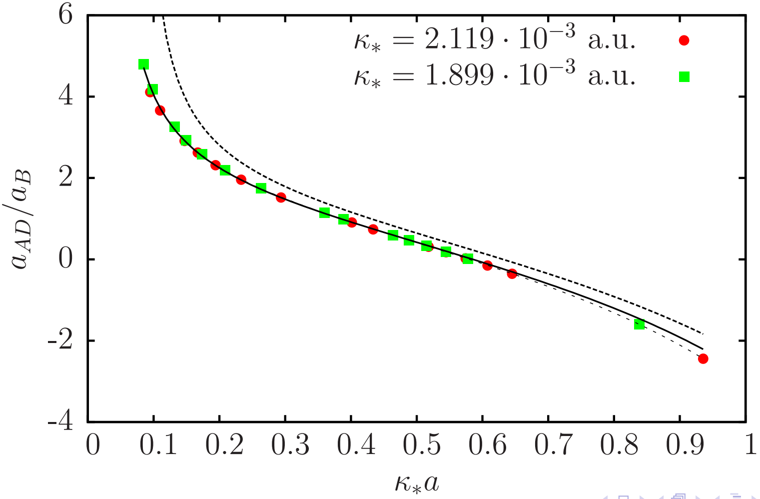

# Universal Effective Range Function

$ka\cot\delta = c_1(ka) + c_2(ka)\cot[s_0\ln(\kappa_* a) + \phi(ka)]$
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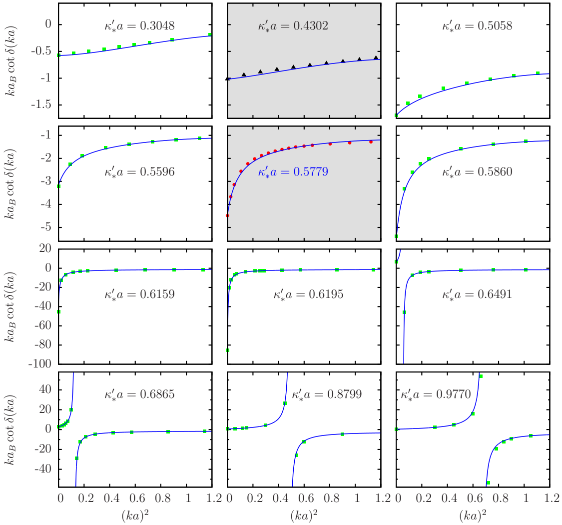


# Low Energy n-d scattering

## Effective range formula

$$k \cot\delta = \frac{-\frac{1}{a_{nd}} + \frac{1}{2} r_s k^2}{1 + k^2/k_0^2}$$

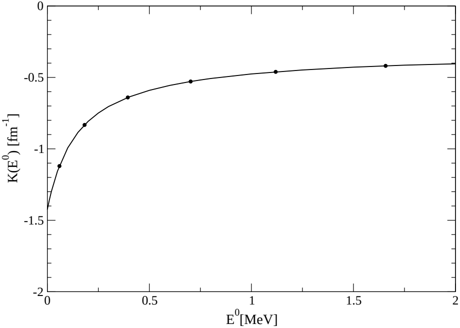
$$E_v = \frac{3}{4}(\hbar^2/m)k_0^2$$

$$a_{nd} \approx 0.7 \text{ fm}$$

$$E_v \approx 160 \text{ keV}$$

$$r_s \approx -127 \text{ fm}$$

C.R. Chen et al., PRC39, 1261 (1989)



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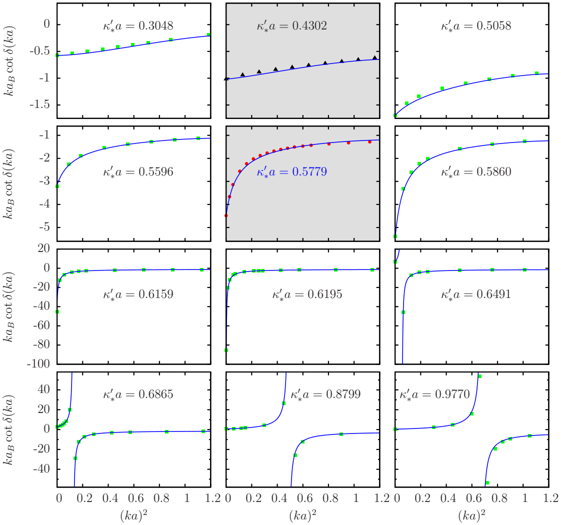


# The NN Volkov's Potential (Soft Potential)

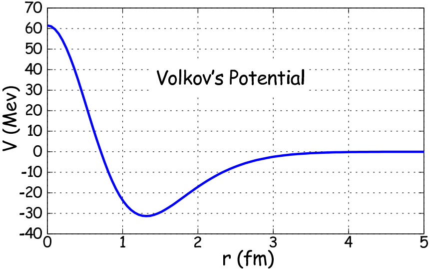
- Mass parameter

$$\hbar^2/m = 41.47\ \mathrm{Mev\,fm}^2$$

- Potential

$$V(r) = E_1\,\mathrm{e}^{-r^2/R_1^2} + E_2\,\mathrm{e}^{-r^2/R_2^2}$$

- $E_1 = 144.86$ Mev, $R_1 = 0.82$ fm, $E_2 = -83.34$ Mev, $R_2 = 1.6$ fm

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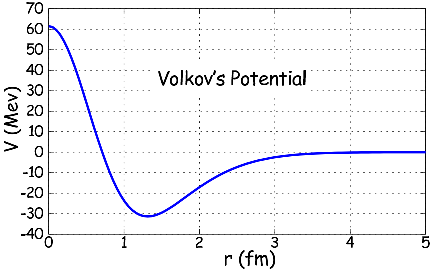



- S-wave potential – only acts when $l_{ij} = 0$

# All-waves Volkov for $A = 6$, $L^\pi = 0^+$

| $K_{max}$ | $N_{HH}$ | $E_0$ (MeV) [6] | $E_1$ (MeV) [6] | $E_2$ (MeV) [5 1] | $E_3$ (MeV) [4 2] |
|---|---|---|---|---|---|
| 2 | 15 | 117.205 | 64.701 | 62.513 | 61.142 |
| 4 | 120 | 118.861 | 69.450 | 64.277 | 62.015 |
| 6 | 680 | 120.345 | 70.544 | 66.268 | 63.377 |
| 8 | 3045 | 121.738 | 71.443 | 67.280 | 64.437 |
| 10 | 11427 | 122.317 | 71.923 | 68.371 | 65.354 |
| 12 | 37310 | 122.597 | 72.477 | 69.029 | 65.886 |
| 14 | 108810 | 122.711 | 72.822 | 69.531 | 66.201 |
| 16 | 288990 | 122.752 | 73.101 | 69.842 | 66.360 |
| 18 | 709410 | 122.768 | 73.284 | 70.051 | 66.437 |
| 20 | 1628328 | 122.774 | 73.407 | 70.189 | 66.474 |
| 22 | 3527160 | 122.776 | 73.485 | 70.283 | 66.491 |
| SVM* | | | | | 66.25 |

* K. Varga and Y. Suzuki, Phys. Rev. C **52**, 2885 (1995)

# All-waves Volkov – Summary

0.546 MeV [2] $0^+$

$A = 2$

0.599 MeV [3] $0^+$

8.465 MeV [3] $0^+$

$A = 3$

8.562 MeV [4] $0^+$

10.406 MeV [3 1] $1^-$

30.418 MeV [4] $0^+$

$A = 4$

28.72 MeV [4 1] $0^+$

31.72 MeV [5] $0^+$

43.03 MeV [4 1] $1^-$

68.28 MeV [5] $0^+$

$A = 5$

66.49 MeV [4 2] $0^+$

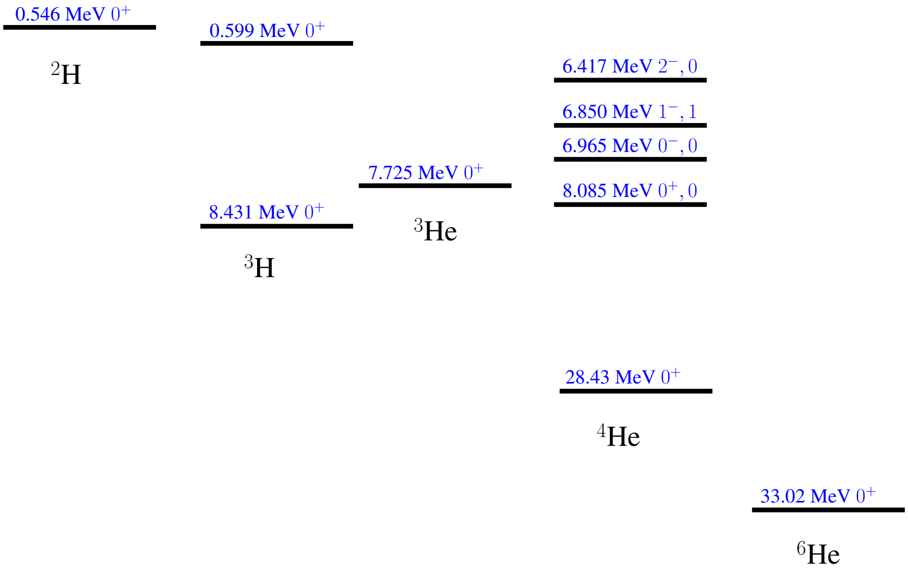
70.28 MeV [5 1] $0^+$

73.49 MeV [6] $0^+$

122.78 MeV [6] $0^+$

$A = 6$

# S-wave Volkov – "Physics"



# Conclusions

## Bosons systems with large scattering length

- natural ($E \approx \hbar^2/mr_0^2$) vs unnatural ($E \approx \hbar^2/ma^2$) energy scale
- The particles are far to each other
- Insensitivity to the short-range part of the interaction
- The spectrum is governed by two parameters: $a, \kappa_*$
- The systems present universal behavior

## Shallow states with Potential models

- Use of soft two- and three-body potentials
- Intensity of the force fixed to reproduce some experimental data
- Treatment of finite range corrections
- Halo nuclei $\leftrightarrow$ Efimov physics

## Evolution with $N$

- General solution of the $N$-boson problem with contact interactions

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## The DSI imposes that

$$E_N^0 + \frac{\hbar^2}{ma^2} = \exp\left[\Delta(\xi)/s_0\right]\frac{\hbar^2\kappa_N^2}{m}$$

## Linear dependence of $\kappa_N$

$$\kappa_N/\kappa_3 = 1 + \gamma(N-3)$$

$\gamma = 1.147$ is an universal quantity (A. Deltuva, FBS 54, 569 (2013)).

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$$\frac{E_N}{E_3} = \left[1 + 1.147(N-3)\right]^2 \approx (1.15N - 2.44)^2$$

## Preliminary result for the $N$-boson energy at unitary limit

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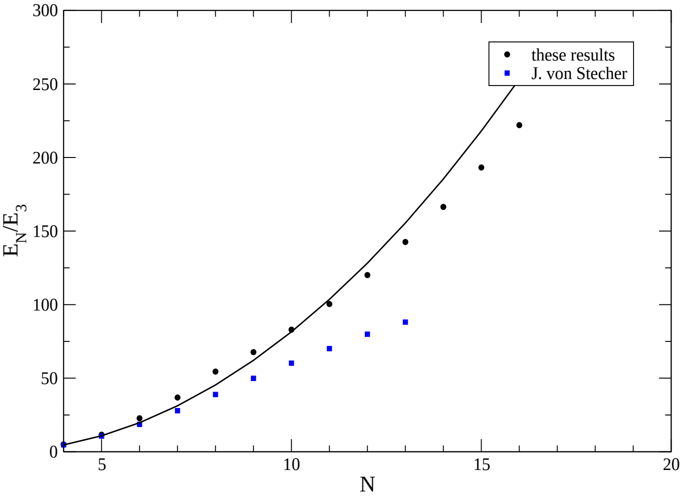
# Testing the $E_N/E_3$ formula at $\xi = -\pi/2$



# Testing the linear dependence on $\kappa_N$ at constant $a$

$$E_N/E_2 = \tan^2\xi$$

$$\kappa_N a_B + \Gamma_N = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

M. Lewerenz, J. Chem. Phys. 106, 4596 (1997) for the TTY potential:

| | TTY | TBG |
|---|---|---|
| $a_0$ (a.u.) | 189.0 | 189.4 |
| $r_0$ (a.u.) | 13.94 | 13.84 |
| $E_2$ (mK) | -1.310 | -1.303 |
| $E_3$ (mK) | -126 | -151 |
| $E_4$ (mK) | -558 | -751 |
| $E_5$ (mK) | -1302 | -1945 |
| . | . | . |
| . | . | . |