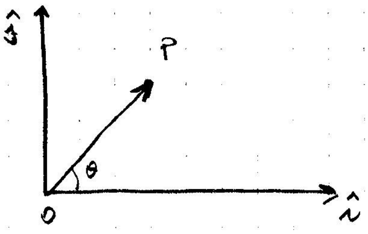
Esercizio del moto in coord. polari

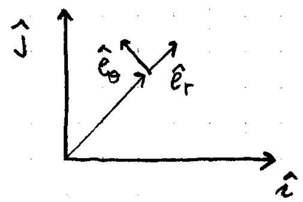
Posizione di P (x, y)



$$\vec{OP} = x\,\hat{i} + y\,\hat{j} = \vec{r} \qquad (1)$$

Dato r e θ
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad (2)$$

Introduco $\hat{e}_r$ e $\hat{e}_\theta$



$$\vec{r} = r\,\hat{e}_r \qquad (3)$$

■ Dato (1) e (3), verifichiamo (2):

$$\begin{cases} \hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \\ \hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \end{cases}$$
[Verifica che sono $\perp$]

$$\Rightarrow \quad \vec{r} = \underbrace{r\cos\theta}_{x}\,\hat{i} + \underbrace{r\sin\theta}_{y}\,\hat{j}$$ ■

<u>Velocità</u>: $\dot{\vec{r}} \equiv \vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$ $\qquad$ $\hat{i}$ e $\hat{j}$ fissi

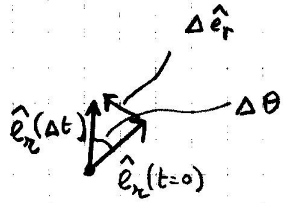
$$\dot{\vec{r}} = \vec{v} = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\hat{i} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\hat{j}$$

$$= \dot{r}\underbrace{\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)}_{\hat{e}_r} + r\dot{\theta}\underbrace{\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)}_{\hat{e}_\theta}$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \qquad (4)$$

■ Verificação da $\vec{r} = r\,\hat{e}_r$

$$\dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\,\frac{d\hat{e}_r}{dt}$$



$$\frac{d\hat{e}_r}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\hat{e}_r}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}\,\hat{e}_\theta = \dot{\theta}\,\hat{e}_\theta$$

$$\Rightarrow \dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \qquad ■$$

<u>Acc</u> : $\dot{\vec{v}} = \vec{a} = \ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath} = \ddot{\vec{r}}$

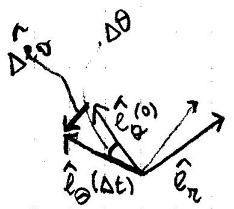
$$= \hat{\imath}\left(\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - \cancel{\dot{r}\dot{\theta}\sin\theta} - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta\right)$$
$$+ \hat{\jmath}\left(\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + \cancel{\dot{r}\dot{\theta}\cos\theta} + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta\right)$$

$$\begin{aligned}
= \;& \ddot{r}\left(\hat{\imath}\cos\theta + \hat{\jmath}\sin\theta\right) \\
&+ 2\dot{r}\dot{\theta}\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right) \\
&+ r\ddot{\theta}\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right) \\
&- r\dot{\theta}^2\left(\cos\theta\,\hat{\imath} + \hat{\jmath}\sin\theta\right)
\end{aligned}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta \qquad (5)$$

■ Verificação da $\dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$

$$\ddot{\vec{r}} = \ddot{r}\,\hat{e}_r + \dot{r}\,\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\frac{d\hat{e}_\theta}{dt}$$



$$\frac{d\hat{e}_\theta}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\hat{e}_\theta}{\Delta t} = \lim_{\Delta t\to 0} -\hat{e}_r\,\frac{\Delta\theta}{\Delta t} =$$
$$= -\hat{e}_r\,\dot{\theta}$$

$$\ddot{\vec{r}} = \ddot{r}\,\hat{e}_r + 2\dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta \qquad ■$$

Due casi particolari: $\dot{r} = 0$ e $\dot{\theta} = 0$

1) $\dot{\theta} = 0 \Rightarrow \vec{v} = \dot{r}\,\hat{e}_r$ $\longrightarrow$ moto rettilineo

$\vec{a} = \ddot{r}\,\hat{e}_r$

2) $\dot{r} = 0 \Rightarrow \begin{cases} \vec{v} = r\dot{\theta}\,\hat{e}_\theta \\ \vec{a} = -r\dot{\theta}^2\,\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta \end{cases}$ $\quad |\vec{v}| = r\dot{\theta}$

$$\begin{aligned} \vec{a} &= \hat{e}_r\left(-\frac{v^2}{r}\right) + \frac{dv}{dt}\,\hat{e}_\theta \\ &= \frac{dv}{dt}\,\hat{e}_\theta - \frac{v^2}{r}\,\hat{e}_r \\ &= \frac{dv}{dt}\,\hat{T} + \frac{v^2}{r}\,\hat{N} \end{aligned}$$