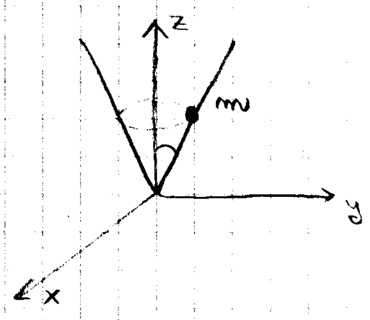
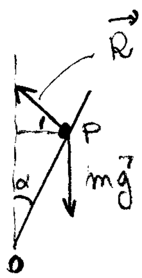
Esercizio della particella in un cono



$\alpha \equiv$ semi apertura

Trovare le eqn del moto: usiamo coord. cilindriche



$$\begin{cases} m\ddot{z} = -mg + R\sin\alpha \\ m(\ddot{r} - r\dot{\theta}^2) = -R\cos\alpha \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \end{cases}$$

Nota che $\frac{r}{z} = \operatorname{tg}\alpha$ $\Rightarrow$

$$(1)\begin{cases} m\ddot{r} = -mg\operatorname{tg}\alpha + R\sin\alpha\operatorname{tg}\alpha \\ m(\ddot{r} - r\dot{\theta}^2) = -R\cos\alpha \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \end{cases}$$

Altro modo di ricavare le eqn. del moto:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + mgz$$

$$\vec{L}_o = m(\vec{r}_o \times \vec{v}_P) = m\begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{z} \\ r & 0 & z \\ \dot{r} & r\dot{\theta} & \dot{z} \end{vmatrix}$$

$$= m\left[\hat{e}_r(-zr\dot{\theta}) - \hat{e}_\theta(r\dot{z} - z\dot{r}) + \hat{z}\, r^2\dot{\theta}\right]$$

$$\vec{M}_o = \vec{r}_o \times \vec{R} + \vec{r}_o \times m\vec{g} = -\sqrt{r^2+z^2}\, R\,\hat{e}_\theta + \sqrt{r^2+z^2}\, mg\sin\alpha\,\hat{e}_\theta$$

$$= \sqrt{r^2+z^2}\,(mg\sin\alpha - R)\,\hat{e}_\theta$$

$$\frac{d\bar{L}_o}{dt} = \bar{M}_o \Rightarrow$$

$$m\Big[\left(-\dot{z}r\dot{\theta} - z\dot{r}\dot{\theta} - zr\ddot{\theta}\right)\hat{e}_r - zr\dot{\theta}\,\overbrace{\hat{e}_\theta\,\dot{\theta}}^{\frac{d\hat{e}_r}{dt} = \dot{\theta}\hat{e}_\theta}$$

$$-\left(\cancel{\dot{r}\dot{z}} + r\ddot{z} - \cancel{\dot{z}\dot{r}} - z\ddot{r}\right)\hat{e}_\theta - \left(r\dot{z} - z\dot{r}\right)\left(-\dot{\theta}\hat{e}_r\right)$$

$$+\hat{z}\left(2\dot{r}r\dot{\theta} + r^2\ddot{\theta}\right)\Big]$$

$$= \sqrt{r^2+z^2}\,\left(mg\sin\alpha - R\right)\hat{e}_\theta$$

$$\Rightarrow \begin{cases} 2\cancel{r}\dot{r}\dot{\theta} + r^{\cancel{2}}\ddot{\theta} = 0 \\ \cancel{r}\dot{z}\dot{\theta} + \cancel{z\dot{r}\dot{\theta}} + z\cancel{r}\ddot{\theta} + \cancel{r}\dot{z}\dot{\theta} - \cancel{\dot{r}\dot{\theta}z} = 0 \\ m\left(-zr\dot{\theta}^2 - r\ddot{z} + z\ddot{r}\right) = \sqrt{r^2+z^2}\,\left(mg\sin\alpha - R\right) \end{cases}$$

$$\Rightarrow \boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0} \qquad (2)$$

$$z\ddot{\theta} + 2\dot{z}\dot{\theta} = 0 = \frac{r\ddot{\theta} + 2\dot{r}\dot{\theta}}{tg\alpha} \Rightarrow \text{non dà altra informazione}$$

$$m\left(-\frac{r^{\cancel{2}}\dot{\theta}^2}{tg\alpha} - \cancel{\frac{r\ddot{r}}{tg\alpha}} + \cancel{\frac{r\ddot{r}}{tg\alpha}}\right) = \cancel{r}\sqrt{1+\frac{1}{tg^2\alpha}}\,\left(mg\sin\alpha - R\right)$$

$$-mr\dot{\theta}^2\frac{\cos\alpha}{\cancel{\sin\alpha}} = \sqrt{1+\frac{\cos^2\alpha}{\sin^2\alpha}}\,\left(mg\sin\alpha - R\right) = \frac{1}{\cancel{\sin\alpha}}\left(mg\sin\alpha - R\right)$$

$$\boxed{mr\dot{\theta}^2\cos\alpha = R - mg\sin\alpha} \qquad (3)$$

La III eq. dalla conservazione dell'energia:

$$\frac{dE}{dt} = 0 = \cancel{\tfrac{1}{2}}\cancel{m}\left(\cancel{2}\dot{r}\ddot{r} + \cancel{2}r\dot{r}\dot{\theta}^2 + \cancel{2}r^2\dot{\theta}\ddot{\theta} + \cancel{2}\dot{z}\ddot{z}\right) + \cancel{m}g\dot{z}$$

$$\boxed{\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta} + \frac{\dot{r}\ddot{r}}{tg^2\alpha} + \frac{g\dot{r}}{tg\alpha} = 0} \qquad (4)$$

(2) – (4) sembrano diverse da (1), però:

Da (2) so che $r\ddot{\theta} = -2\dot{r}\dot{\theta}$ $\Rightarrow$ (4) diventa

$$\cancel{\dot{r}}\ddot{r} + r\cancel{\dot{r}}\dot{\theta}^2 + r\dot{\theta}(-2\cancel{\dot{r}}\dot{\theta}) + \frac{\ddot{r}\cancel{\dot{r}}}{tg^2\alpha} + \frac{g\cancel{\dot{r}}}{tg\alpha} = 0$$

$$\ddot{r} - r\dot{\theta}^2 + \frac{\ddot{r}}{tg^2\alpha} + \frac{g}{tg\alpha} = 0$$

$$\ddot{r} - r\dot{\theta}^2 + \ddot{r}\frac{\cos^2\alpha}{\sin^2\alpha} + g\frac{\cos\alpha}{\sin\alpha} = 0$$

$$\frac{\ddot{r}}{\sin^2\alpha} + g\frac{\cos\alpha}{\sin\alpha} - r\dot{\theta}^2 = 0$$

$$m\ddot{r}\frac{\cos\alpha}{\sin^2\alpha} + mg\frac{\cos^2\alpha}{\sin\alpha} = mr\dot{\theta}^2\cos\alpha \overset{(3)}{=} R - mg\sin\alpha$$

$$m\ddot{r} = R\frac{\sin^2\alpha}{\cos\alpha} - mg\frac{\sin\alpha}{\cos\alpha}$$ $\Rightarrow$ è uguale alla I eq. di (1)

Combinando queste eqn. con (3)

$$m\ddot{r} - mr\dot{\theta}^2 = R\frac{\sin^2\alpha}{\cos\alpha} - \cancel{mg\frac{\sin\alpha}{\cos\alpha}} - \frac{R}{\cos\alpha} + \cancel{mg\frac{\sin\alpha}{\cos\alpha}}$$

$$= R\frac{-(1-\sin^2\alpha)}{\cos\alpha} = -R\cos\alpha$$

$\Downarrow$

è la II eqn. di (1) !

Quanto deve essere $\dot\theta$ per avere un moto circolare?

Moto circ. $\Rightarrow$ $\dot r = \ddot r = 0$ $\Rightarrow$ (1) diventa

$$\begin{cases} -m r\dot\theta^2 = -R\cos\alpha \\ mg = R\sin\alpha \end{cases} \Rightarrow \quad \cancel{m} r\dot\theta^2 = \cancel{m} g \frac{\cos\alpha}{\sin\alpha}$$

$$\dot\theta = \sqrt{\frac{g}{r}\frac{\cos\alpha}{\sin\alpha}}$$

Trovare r_{min} e r_{max} (si potrà avere $r_{min} = 0$?)

$$E = \frac{1}{2} m \dot r^2 \left(1 + \frac{1}{tg^2\alpha}\right) + \frac{1}{2} m r^2 \dot\theta^2 + mg\frac{r}{tg\,\alpha}$$

$L_{o_z} = m r^2 \dot\theta$ ed è costante $\Rightarrow$

$$E = \frac{1}{2} m \frac{\dot r^2}{\sin^2\alpha} + \frac{1}{2} \cancel{m} \cancel{r^2} \frac{L_{o_z}^2}{m^{\cancel{2}} r^{\cancel{4}2}} + mg\,r\frac{\cos\alpha}{\sin\alpha}$$

$$= \frac{1}{2} m \left(\frac{\dot r}{\sin\alpha}\right)^2 + \frac{L_{o_z}^2}{2mr^2} + mg\,r\frac{\cos\alpha}{\sin\alpha} = \text{cost.}$$

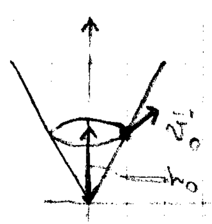
Se $r = 0$ $\Rightarrow$ $E = \infty$ $\Rightarrow$ non potrò mai avere $r = 0$

r_{max} e r_{min} $\Rightarrow$ $\dot r = 0$ $\Rightarrow$

$$E = \frac{L_{o_z}^2}{2mr^2} + mgr\frac{\cos\alpha}{\sin\alpha}$$

$$2m^2 g\frac{\cos\alpha}{\sin\alpha} r^3 - 2mr^2 E + L_{o_z}^2 = 0$$

Supponiamo che a $t = 0$



$\vec v_0 = v_0\,\hat e_\theta$

$r_0 = h_0\,tg\alpha$

$\Rightarrow$ $E = \frac{1}{2} m v_0^2 + mgh_0$

$L_{o_z} = m\,h_0\,tg\alpha\, v_0$

$$2 m^2 g r^3 \frac{\cos\alpha}{\sin\alpha} - 2 m r^2 \left(\frac{1}{2} m v_0^2 + m g \frac{r_0}{\sin\alpha} \cos\alpha \right)$$

$$+ \; m^2 r_0^2 \cancel{\frac{\cos^2\alpha}{\sin^2\alpha}} \cancel{\left(\frac{\sin\alpha}{\cos\alpha}\right)^2} v_0^2 = 0$$

$$2 m^2 g \frac{\cos\alpha}{\sin\alpha} r^3 - m^2 v_0^2 (r^2 - r_0^2) - 2 m^2 g r^2 r_0 \frac{\cos\alpha}{\sin\alpha} = 0$$

$$2 \cancel{m^2} g \frac{\cos\alpha}{\sin\alpha} r^2 \cancel{(r - r_0)} - \cancel{m^2} v_0^2 \cancel{(r - r_0)} (r + r_0) = 0$$

$$2 g \frac{\cos\alpha}{\sin\alpha} r^2 - v_0^2 r - v_0^2 r_0 = 0$$

$$r = \frac{+ v_0^2 \pm \sqrt{v_0^4 + 8 g v_0^2 \frac{\cos\alpha}{\sin\alpha} r_0}}{4 g \frac{\cos\alpha}{\sin\alpha}}$$

$$\Rightarrow \begin{cases} r_{min} = \dfrac{v_0^2 \sin\alpha}{4 g \cos\alpha} \left[1 + \sqrt{1 + \dfrac{8 g r_0 \cos\alpha}{v_0^2 \sin\alpha}} \right] \\[2ex] r_{max} = r_0 \end{cases}$$

Esercizio dell'urto di 2 particelle in campo centrale

agisce una forza radiale

$$F(r) = -\frac{K}{r^2} \qquad r: \text{dist. relativa}$$

1) Scriviamo energia potenziale per un generico r

$$V(r) - V(\infty) = -\int_{\infty}^{r} F(r')\, dr'$$

$$= K \int_{\infty}^{r} \frac{1}{r'^2}\, dr' = -\frac{K}{r'}\Bigg|_{\infty}^{r}$$

Imponiamo $V(\infty) = 0 \Rightarrow V(r) = -\frac{K}{r}$

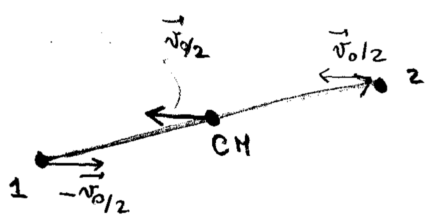
2) Costanti del moto del sistema delle due masse:

$M\vec{v}_{CM}$ (non ci sono forze esterne)

E (forze conservative)

$\vec{J}_{CM}$: mom. angolare rispetto al CM (forze hanno mom. nullo)

$\Rightarrow \vec{v}_{CM} = \frac{m\vec{v}_0}{2m} = \frac{\vec{v}_0}{2}$ e mi metto nel SR del CM



$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\vec{v}_r = \vec{v}_2 - \vec{v}_1$$

Scriviamo l'energia in questo sist. di rif:

$t=0 \qquad E = \frac{1}{2}\mu v_r^2 + \cancel{V(\infty)} = \frac{1}{2}\mu v_0^2 \qquad \mu = \frac{m}{2}$

generico $t \qquad E = \frac{1}{2}\mu v_r^2 + V(r) =$

$$= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{K}{r}$$

Modulo del mom. angolare

espressione generale

$$t=0 \qquad J = \overbrace{m\,\frac{v_0}{2}\,\frac{b}{2}}^{\text{part. 1}} + \overbrace{m\,\frac{v_0}{2}\,\frac{b}{2}}^{\text{part. 2}} = \frac{m}{2}\,v_0 b = \mu\,|\vec{r}\times\vec{v}_r|$$

■ infatti $\vec{J} = m\,\vec{r}_1\times\vec{v}_1 + m\,\vec{r}_2\times\vec{v}_2$

$$\vec{r}_{CM} = \frac{\vec{r}_1+\vec{r}_2}{2} \qquad \vec{r}_1 = \vec{r}_{CM} - \frac{\vec{r}}{2} \qquad \vec{v}_1 = \vec{v}_{CM} - \frac{\vec{v}_r}{2}$$

$$\Rightarrow \qquad \vec{r}_2 = \vec{r}_{CM} + \frac{\vec{r}}{2} \qquad \vec{v}_2 = \vec{v}_{CM} + \frac{\vec{v}_r}{2}$$

$$\vec{v}_{CM} = \frac{\vec{v}_1+\vec{v}_2}{2}$$

$$J = m\Big[\vec{r}_{CM}\times\vec{v}_{CM} - \vec{r}_{CM}\times\frac{\vec{v}_r}{2} - \frac{\vec{r}}{2}\times\vec{v}_{CM} + \frac{\vec{r}}{2}\times\frac{\vec{v}_r}{2}$$

$$+ \vec{r}_{CM}\times\vec{v}_{CM} + \vec{r}_{CM}\times\frac{\vec{v}_r}{2} + \frac{\vec{r}}{2}\times\vec{v}_{CM} + \frac{\vec{r}}{2}\times\frac{\vec{v}_r}{2}\Big]$$

$$= 2m\,\vec{r}_{CM}\times\vec{v}_{CM} + \frac{m}{2}\,\vec{r}\times\vec{v}_r = M_{tot}\,\vec{r}_{CM}\times\vec{v}_{CM} + \mu\,\vec{r}\times\vec{v}_r \;■$$

generico t $\qquad J = \mu r^2\dot{\theta}$

$$\Rightarrow \quad E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - \frac{K}{r}$$

3) r_{min} : $\qquad E = \frac{1}{2}\mu v_0^2 = \underbrace{\frac{J^2}{2\mu r^2} - \frac{K}{r}}_{V_{eff}(r)}$

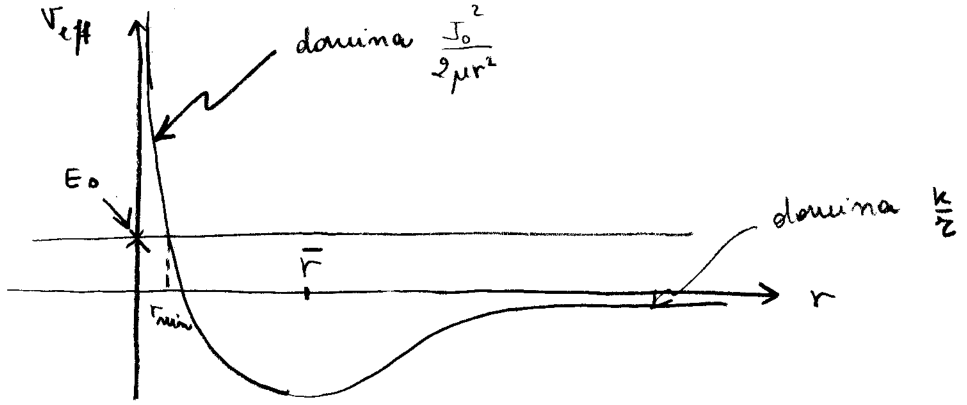
$$\frac{1}{2}\mu v_0^2 = E_0 = \frac{J_0^2}{2\mu r^2} - \frac{K}{r} \qquad J_0 = \mu v_0 b$$

$$2\mu E_0 r^2 + 2\mu k r - J_0^2 = 0$$

$$r = \frac{-\mu k \pm \sqrt{\mu^2 k^2 + 2\mu E_0 J_0^2}}{2\mu E_0}$$

$$r_{min} = -\frac{K}{2E_0} \pm \sqrt{\frac{\cancel{\mu^2} k^2}{4\cancel{\mu^2} E_0^2} + \frac{\cancel{2\mu E_0} J_0^2}{\cancel{4}\mu^2 E_0^{\cancel{2}}}} = +\frac{K}{2E_0}\left[-1 \pm \sqrt{1 + \frac{2E_0 J_0^2}{\mu k^2}}\right]$$

Analisi grafica:



$$\frac{d V_{eff}}{dr} = -\frac{\cancel{2} J_0^2}{\cancel{2}\mu r^3} + \frac{k}{r^2} = 0 \Rightarrow$$

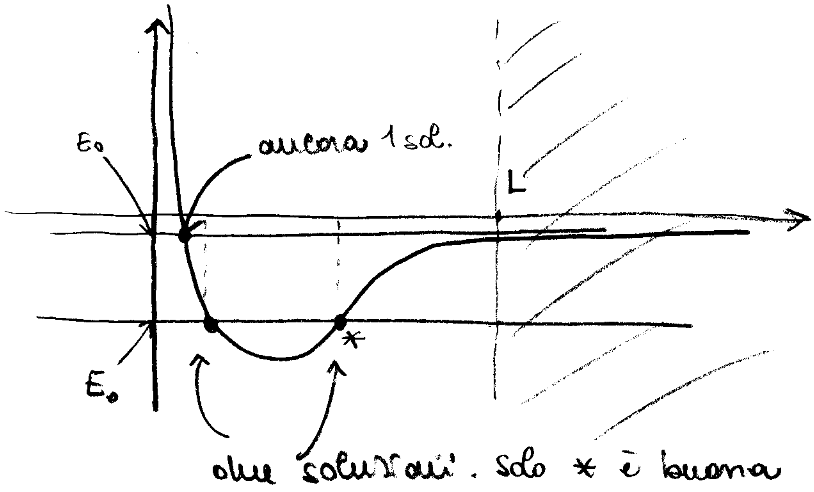
$$\frac{J_0^2}{\mu r} = k \qquad r = \frac{J_0^2}{k\mu} = \frac{\mu^2 b^2 v_0^2}{k\cancel{\mu}} = \frac{\mu}{k} b^2 v_0^2$$

$$\Rightarrow \quad \bar{r} = \frac{\mu}{k} b^2 v_0^2$$

Supponiamo che a $t=0$ $E_0 < 0$ (per esempio $r(t=0) = L$ e $\frac{1}{2}\mu v_0^2 - \frac{k}{L} < 0 \Rightarrow$

$\frac{k}{L} > \frac{1}{2}\mu v_0^2 \qquad L < \frac{2k}{\mu v_0^2}$)

Posso avere 2 casi



dipende dai parametri iniziali