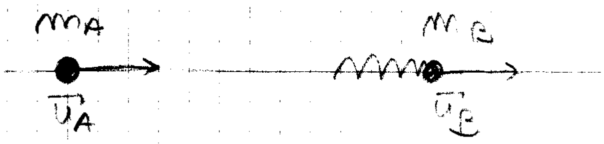
Esercizio dell'urto centrale elastico II



massa della molla ~0

$|\vec{u}_A| > |\vec{u}_B|$

Calcolare $\vec{v}_A$ e $\vec{v}_B$ dopo l'urto:

(i) Risolviamo con le eqn. del moto

$$\begin{cases} M\ddot{x}_{CM} = 0 \\ \mu\ddot{x} = -k(x - l_0) \end{cases}$$

per A in contatto con la molla $x \equiv x_B - x_A$

$$x_{CM}(t) = \underbrace{x_{CM}(0)}_{m_B l_0/M} + v_{CM}\,t \qquad v_{CM} = \frac{m_A u_A + m_B u_B}{M}$$

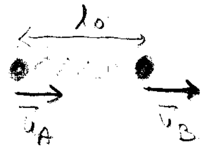
$$x(t) = l_0 + A\sin(\omega t + \varphi) \qquad \omega = \sqrt{\frac{k}{\mu}} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\mu}{k}}$$

$$\begin{aligned} x(0) &= l_0 \\ \dot{x}(0) &= u_B - u_A = u_{rel} \end{aligned} \Rightarrow \begin{aligned} A\sin\varphi &= 0 \\ A\omega\cos\varphi &= v_{rel} \end{aligned} \Rightarrow$$

$$x(t) = l_0 + \frac{u_{rel}}{\omega}\sin\omega t$$

Seguiamo come evolve la situazione:

1) $t = 0$



2) $t = \frac{T}{4} = \frac{\pi}{2\omega}$ $\qquad \dot{x}(t) = \frac{v_{rel}}{\cancel{\omega}}\cancel{\omega}\cos\omega t = u_{rel}\cos\frac{\omega\pi}{2\omega} = 0$

$$\Rightarrow x_{min} = l_0 - \frac{u_{rel}}{\omega}$$

$$\begin{aligned} v_A &= v_{CM} - \frac{m_B}{M}\dot{x}(t) \\ v_B &= v_{CM} + \frac{m_A}{M}\dot{x}(t) \end{aligned} \Rightarrow v_A = v_B$$

3) $t = \frac{T}{2} = \frac{\pi}{\omega}$ $\qquad x(t) = l_0 \Rightarrow$ la pallina A si distacca

$$\dot{x}\left(\frac{T}{2}\right) = \frac{u_{rel}}{\omega}\,\omega\cos\pi = -u_{rel}$$

fine dell'urto!

$$\Rightarrow \quad v_B - v_A = u_A - u_B \quad !$$

$$v_A = v_{CM} + \frac{m_B}{M} u_{rel} = \frac{m_A u_A + m_B u_B + m_B u_B - m_B u_A}{M}$$

$$v_B = v_{CM} - \frac{m_A}{M} u_{rel} = \frac{m_A u_A + m_B u_B - m_A u_B + m_A u_A}{M}$$

$$\begin{cases} v_A = \dfrac{(m_A - m_B)\, u_A + 2 m_B u_B}{M} \\[2ex] v_B = \dfrac{2 m_A u_A + (m_B - m_A)\, u_B}{M} \end{cases}$$

Nota: se $K \to \infty$ $\frac{T}{2} \to 0$ $\Rightarrow$ urto istantaneo

$$I_B = \int_0^{T/2} F_{el} \cdot dt = -k (x - l_0)\, dt$$

$$= -k \frac{u_{rel}}{\omega} \int_0^{T/2} \sin \omega t \, dt = +k \frac{u_{rel}}{\omega^2} \cos \omega t \Big|_0^{T/2}$$

$$= k \frac{u_{rel}}{\omega^2} (-1-1) = -2k \frac{u_{rel}}{\omega^2}$$

$$= -2 \cancel{k}\, u_{rel} \frac{\mu}{\cancel{k}}$$

$$= -2\, u_{rel}\, \mu$$

$I_B = -2 \dfrac{m_A m_B}{M} (u_B - u_A)$. Ovviamente $I_A = -I_B$ e $I_A + I_B = 0$

Dal resto:

$$\Delta Q = m_A u_A + m_B u_B - m_A v_A - m_B v_B =$$

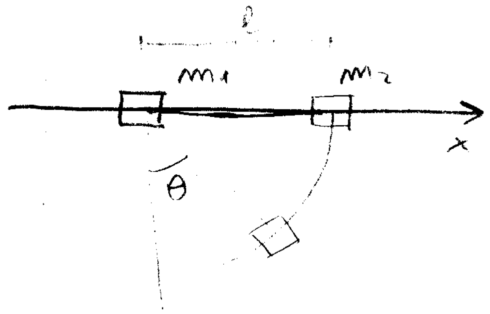
$$= \frac{1}{M} \Big(\cancel{m_A^2 u_A} + m_A m_B u_A + \cancel{m_B^2 u_B} + \cancel{m_A m_B u_B}$$
$$- \cancel{m_A^2 u_A} + m_A m_B u_A - 2 \cancel{m_A m_B u_B}$$
$$- 2 \cancel{m_A m_B u_A} - \cancel{m_B^2 u_B} + \cancel{m_A m_B u_B} \Big) = 0$$

ii) Con le conservazioni: non ci sono $\vec{F}_{ext} \Rightarrow \vec{Q}_{tot} = \text{cost}$

Forze conservative $\Rightarrow$ $E = \text{cost}$

Nota $E_{pot}(t=0) = E_{pot}(T/2) = 0 \Rightarrow$ ricavo come sempre.

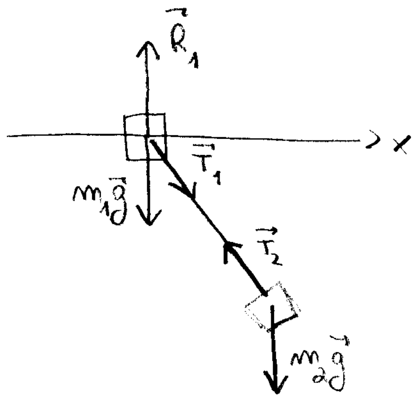
Esercizio del pendolo doppio



m_1 libero di scorrere lungo x senza attrito.

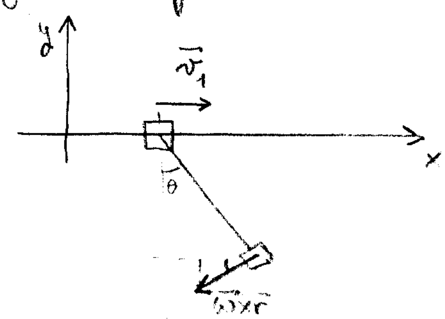
Dato m_1, m_2 e l (lunghezza del filo), trovare $v_1(\theta)$ e $v_2(\theta)$

Forze



Forze esterne $\parallel \vec{y}$ $\Rightarrow$ $m_1 v_1 + m_2 v_{2x} = 0$ (1)

In generale posso scrivere $\vec{v}_2 = \vec{v}_1 + \vec{\omega} \times \vec{r}$



$$\Rightarrow \quad v_{2,x} = v_1 - \omega l \cos\theta$$
$$v_{2y} = - \omega l \sin\theta$$

$$\Rightarrow \quad m_1 v_1 + m_2 v_1 - m_2 \omega l \cos\theta = 0$$

$$\Rightarrow \quad v_1 = \frac{m_2}{m_{12}} \omega l \cos\theta$$

Adesso devo trovare ω:

$$\mathcal{L}_{Forze} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad \leftarrow \text{teorema delle forze vive}$$

$$\mathcal{L}_{R_1} = 0 \quad (\vec{R}_1 \perp d\vec{r}_1)$$

$$\mathcal{L}_{m_1 g} = 0 \quad (m_1\vec{g} \perp d\vec{r}_1)$$

$$\mathcal{L}_{T_1} = \int \vec{T}_1 \cdot d\vec{r}_1 = T_1 \cdot \Delta x_1 \sin\theta$$

$$\mathcal{L}_{T_2} = \int \vec{T}_2 \cdot d\vec{r}_2 = \int \vec{T}_2 \cdot \left(d\vec{r}_1 + d(\vec{\omega} \times \vec{r}_2) \right) \underset{\uparrow}{=} \int \vec{T}_2 \cdot d\vec{r}_1 = - \mathcal{L}_{T_1}$$

$d(\vec{\omega} \times \vec{r}_2) \perp$ filo

$$\Rightarrow \mathcal{L}_{\text{Forze}} = \mathcal{L}_{m_2 g} = m_2 g \, l \cos\theta$$

$$m_2 g l \cos\theta = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 \left(v_1^2 + \omega^2 l^2 - 2 v_1 \omega l \cos\theta \right)$$

$$= \cancel{\frac{1}{2} m_1 \frac{m_2^2}{m_{12}^2} \omega^2 l^2 \cos^2\theta} + \frac{1}{2} m_2 \omega^2 l^2$$

$$- m_2 \frac{m_2}{m_{12}} \omega^2 l^2 \cos^2\theta \; \frac{1}{2}$$

$$= \frac{1}{2} m_2 \omega^2 l^2 \left(1 - \frac{m_2}{m_{12}} \cos^2\theta \right)$$

$$= \frac{1}{2} m_2 \omega^2 l^2 \left(\frac{m_1 + m_2 \sin^2\theta}{m_{12}} \right)$$

$$\Rightarrow \omega = \sqrt{2 \frac{g}{l} \cos\theta \frac{m_1 + m_2}{m_1 + m_2 \sin^2\theta}} \, .$$

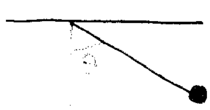
Consideriamo $m_1 \to \infty$:

$$\omega \to \sqrt{2 \frac{g}{l} \cos\theta}$$

$$|\vec{v}_1| \to 0$$

$$|\vec{v}_2| \to \sqrt{2 g l \cos\theta}$$

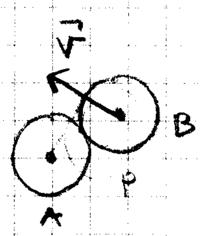
Questo caso corrisponde al pendolo semplice:



$$\mathcal{L} = m g l \cos\theta = \frac{1}{2} m v^2$$

$$\Rightarrow \sqrt{2 g l \cos\theta} = |\vec{v}_2| \; !$$

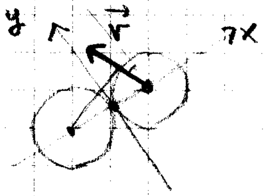
Esercizio sull'urto di 2 sfere rigide



r, m, P

Non c'è attrito di sorta $\Rightarrow$ le forze esercitate nell'urto sono lungo la congiungente dei centri

$\Downarrow$



$$\begin{cases} V_y = u_{B,y} \quad e \quad u_{Ay} = 0 \\ m_B V_x = m_B u_{B,x} + m_A u_{A,x} \\ m_B(V_x^2 + \cancel{V_y^2}) = m_B(u_{B,x}^2 + \cancel{u_{B,y}^2}) + m_A u_{A,x}^2 \end{cases}$$

$$\Rightarrow \begin{cases} m_B V_x = m_B u_{Bx} + m_A u_{Ax} \\ m_B V_x^2 = m_B u_{B,x}^2 + m_A u_{Ax}^2 \end{cases}$$

$$u_{Ax} = \frac{m_B}{m_A}(V_x - u_{Bx})$$

$$\cancel{m_B} V_x^2 = \cancel{m_B} u_{Bx}^2 + \cancel{m_A} \frac{m_B^{\cancel{2}}}{m_A^{\cancel{2}}} V_x^2 + \cancel{m_A} \frac{m_B^{\cancel{2}}}{m_A^{\cancel{2}}} u_{B,x}^2 - 2 \frac{m_B^{\cancel{2}}}{m_A^{\cancel{2}}} V_x u_{Bx} \cancel{m_A}$$

$$u_{Bx}^2 \frac{M_T}{\cancel{m_A}} - 2 \frac{m_B}{\cancel{m_A}} V_x u_{Bx} + \frac{m_B - m_A}{\cancel{m_A}} V_x^2 = 0$$

$$u_{B,x} = \frac{m_B V_x \pm \sqrt{\cancel{m_B^2} V_x^2 - (\cancel{m_B^2} - m_A^2) V_x^2}}{m_A + m_B}$$

urto non c'è stato!

$$= V_x \frac{m_B + m_A}{m_A + m_B} = V_x \frac{m_B - m_A}{m_A + m_B}$$

$$u_{Ax} = \frac{m_B}{m_A} v_x \left(1 - \frac{m_B - m_A}{m_A + m_B}\right) = \frac{m_B}{m_A} v_x \quad \frac{2 m_A}{m_A + m_B}$$

$$= \frac{2 m_B}{m_A + m_B} \quad v_x$$

Vediamo gli angoli tra $\vec{u}_A$, $\vec{u}_B$ e $\vec{v}$:

$$\vec{u}_A \cdot \vec{u}_B = u_{Ax} u_{Bx} + u_{Ay} u_{By} = 2 m_B \frac{m_B - m_A}{(m_A + m_B)^2} v_x^2$$

$$= u_A u_B \cos\theta_{AB}$$

$$\cos\theta_{AB} = \frac{u_{Bx}}{u_B} = \frac{m_B - m_A}{m_A + m_B} \frac{v_x}{\sqrt{v_x^2 \left(\frac{m_B - m_A}{m_A + m_B}\right)^2 + v_y^2}}$$

$$\vec{u}_B \cdot \vec{v} = v_y^2 + u_{Bx} v_x = v_y^2 + v_x^2 \frac{m_B - m_A}{m_A + m_B}$$

$$\cos\theta_B = \frac{v_x^2 \frac{m_B - m_A}{m_A + m_B} + v_y^2}{v \sqrt{v_x^2 \left(\frac{m_B - m_A}{m_A + m_B}\right)^2 + v_y^2}}$$

Caso interessante è $m_A = m_B$:

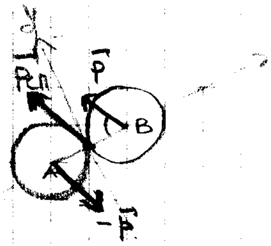
$$\begin{cases} \cos\theta_{AB} = 0 & \theta_{AB} = \frac{\pi}{2} \\ \cos\theta_B = \frac{v_y}{v} = & = \frac{b}{2r} \\ u_{Bx} = 0 & u_{By} = v\left(\frac{b}{2r}\right) \\ u_{Ax} = v_x & u_{Ay} = 0 \end{cases}$$

SR c.m.

$$\begin{cases} \vec{P}_{CM} = \vec{P}_B \\ \vec{p} = \dfrac{m_A \vec{P}_B - m_B \vec{P}_A}{m_A + m_B} \end{cases} \qquad \vec{q} = \frac{m_A \vec{q}_B - m_B \vec{q}_A}{m_A + m_B}$$

$$\Downarrow$$

$$\begin{cases} \vec{P}_A = \dfrac{m_A}{M_T} P_{CM} - \vec{p} \\ \vec{P}_B = \dfrac{m_B}{M_T} P_{CM} + \vec{p} \end{cases} \qquad \begin{aligned} \vec{q}_A &= \frac{m_A}{M_T} P_{CM} - \vec{q} \\ \vec{q}_B &= \frac{m_B}{M_T} P_{CM} + \vec{q} \end{aligned}$$



Non agiscono forze lungo y né su A, né su B $\Rightarrow$

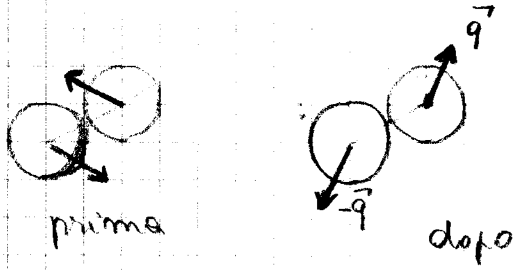
$$q_y = p_y$$

Nel S.R. CM c. energia cinetica:

$$\cancel{\frac{P_{CM}^2}{2M_T}} + \frac{p^2}{2\mu} = \cancel{\frac{P_{CM}^2}{2M_T}} + \frac{q^2}{2\mu}$$

$$p_x^2 + \cancel{p_y^2} = q_x^2 + \cancel{q_y^2} \qquad q_x = \pm p_x$$

$\Rightarrow$ $q_x = -p_x$ (altrimenti non ho urto!)



Verifichiamo con

$$\vec{q} = \vec{q}_B - \frac{m_B}{m_T}\vec{P}_B$$

$$q_x = m_B u_{B,x} - \frac{m_B}{m_T} m_B V_x$$

$$= m_B V_x \left(\frac{\cancel{m_B} - m_A}{m_T} - \cancel{\frac{m_B}{m_T}} \right) = -\frac{m_A m_B}{m_T} V_x$$

$$p_x = m_B V_x - \frac{m_B}{m_T} m_B V_x = \frac{m_B m_A V_x}{m_T}$$

$\Rightarrow$ $q_x = -p_x$ (nota che $p_x = \mu V_x$!)

$$q_y = m_B u_{B\,x} - \frac{m_B}{m_T} m_B V_y =$$

$$= m_B V_y \left(1 - \frac{m_B}{m_T}\right) = \mu V_y = p_y$$

Angolo tra $\vec{p}$ e $\vec{q}$

$$\cancel{pq}\cos\theta = p_x q_x + p_y q_y = p_x^2 - p_y^2$$

$$= \cancel{p^2}\cos^2\alpha - \cancel{p}\sin^2\alpha$$

$$\cos\theta = (1 - 2\sin^2\alpha) = 1 - \cancel{2}\frac{p^2}{\cancel{4}_2 r^2} = \frac{2r^2 - p^2}{2r^2}.$$

$\Rightarrow$ nel c.M. l'angolo dipende da r e p, ma non dalle masse!