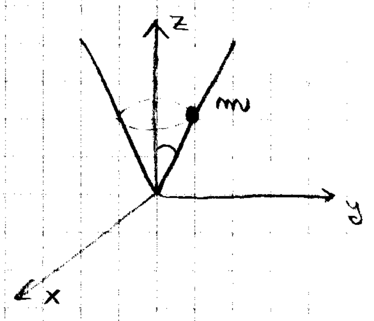
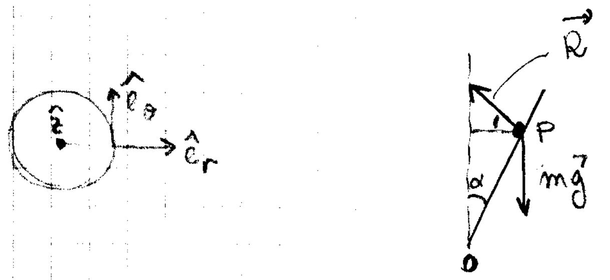
Esercizio della particella in un cono



$\alpha \equiv$ semiapertura

Trovare le eqn del moto: usiamo coord. cilindriche



$$\begin{cases} m\ddot{z} = -mg + R\sin\alpha \\ m(\ddot{r} - r\dot{\theta}^2) = -R\cos\alpha \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \end{cases}$$

Nota che $\frac{r}{z} = \mathrm{tg}\,\alpha \Rightarrow$

$$(1)\begin{cases} m\ddot{r} = -mg\,\mathrm{tg}\alpha + R\sin\alpha\,\mathrm{tg}\alpha \\ m(\ddot{r} - r\dot{\theta}^2) = -R\cos\alpha \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \end{cases}$$

Altro modo di ricavare le eqn. del moto:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + mgz$$

$$\vec{L}_o = m(\vec{r}_o \times \vec{v}_P) = m\begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{z} \\ r & 0 & z \\ \dot{r} & r\dot{\theta} & \dot{z} \end{vmatrix}$$

$$= m\left[\hat{e}_r(-z r\dot{\theta}) - \hat{e}_\theta(r\dot{z} - z\dot{r}) + \hat{z}\, r^2\dot{\theta}\right]$$

$$\vec{M}_o = \vec{r}_o \times \vec{R} + \vec{r}_o \times m\vec{g} = -\sqrt{r^2+z^2}\,R\,\hat{e}_\theta + \sqrt{r^2+z^2}\,mg\sin\alpha\,\hat{e}_\theta$$
$$= \sqrt{r^2+z^2}\,(mg\sin\alpha - R)\,\hat{e}_\theta$$

$$\frac{d\vec{L}_0}{dt} = \vec{M}_0 \Rightarrow$$

$$m\Big[\left(-\dot{z}r\dot{\theta} - z\dot{r}\dot{\theta} - zr\ddot{\theta}\right)\hat{e}_r - zr\dot{\theta}\,\overbrace{\hat{e}_\theta\,\dot{\theta}}^{\frac{d\hat{e}_r}{dt} = \dot{\theta}\hat{e}_\theta}$$

$$-\left(\cancel{\dot{r}\dot{z}} + r\ddot{z} - \cancel{\dot{z}\dot{r}} - z\ddot{r}\right)\hat{e}_\theta - \left(r\dot{z} - z\dot{r}\right)\left(-\dot{\theta}\hat{e}_r\right)$$

$$+\hat{z}\left(2\dot{r}r\dot{\theta} + r^2\ddot{\theta}\right)\Big]$$

$$= \sqrt{r^2+z^2}\,(mg\sin\alpha - R)\,\hat{e}_\theta$$

$$\Rightarrow \begin{cases} 2\cancel{r}\dot{r}\dot{\theta} + r^{\cancel{2}}\ddot{\theta} = 0 \\ \cancel{r}\dot{z}\dot{\theta} + \cancel{z\dot{r}\dot{\theta}} + z\cancel{r}\ddot{\theta} + \cancel{r}\dot{z}\dot{\theta} - \cancel{\dot{r}\dot{\theta}z} = 0 \\ m\left(-zr\dot{\theta}^2 - r\ddot{z} + z\ddot{r}\right) = \sqrt{r^2+z^2}\,(mg\sin\alpha - R) \end{cases}$$

$$\Rightarrow \boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0} \qquad (2)$$

$$z\ddot{\theta} + 2\dot{z}\dot{\theta} = 0 = \frac{r\ddot{\theta} + 2\dot{r}\dot{\theta}}{\mathrm{tg}\,\alpha} \Rightarrow$$ non dà altra informazione

$$m\left(-\frac{r^{\cancel{2}}\dot{\theta}^2}{\mathrm{tg}\,\alpha} - \cancel{\frac{r\ddot{r}}{\mathrm{tg}\,\alpha}} + \cancel{\frac{r\ddot{r}}{\mathrm{tg}\,\alpha}}\right) = \cancel{r}\sqrt{1+\frac{1}{\mathrm{tg}^2\alpha}}\,(mg\sin\alpha - R)$$

$$-mr\dot{\theta}^2\frac{\cos\alpha}{\cancel{\sin\alpha}} = \sqrt{1+\frac{\cos^2\alpha}{\sin^2\alpha}}\,(mg\sin\alpha - R) = \frac{1}{\cancel{\sin\alpha}}(mg\sin\alpha - R)$$

$$\boxed{mr\dot{\theta}^2\cos\alpha = R - mg\sin\alpha} \qquad (3)$$

La III eq. dalla conservazione dell'energia:

$$\frac{dE}{dt} = 0 = \cancel{\frac{1}{2}}\cancel{m}\left(\cancel{2}\dot{r}\ddot{r} + \cancel{2}r\dot{r}\dot{\theta}^2 + \cancel{2}r^2\dot{\theta}\ddot{\theta} + \cancel{2}\dot{z}\ddot{z}\right) + \cancel{m}g\dot{z}$$

$$\boxed{\dot r \ddot r + r \dot r \dot\theta^2 + r^2 \dot\theta \ddot\theta + \frac{\dot r \ddot r}{tg^2\alpha} + \frac{g \dot r}{tg\alpha} = 0} \qquad (4)$$

(2) – (4) sembrano diverse da (1), però:

Da (2) so che $r\ddot\theta = -2\dot r\dot\theta$ $\Rightarrow$ (4) diventa

$$\cancel{\dot r}\,\ddot r + r\cancel{\dot r}\,\dot\theta^2 + r\dot\theta(-2\cancel{\dot r}\dot\theta) + \frac{\ddot r \cancel{\dot r}}{tg^2\alpha} + \frac{g\cancel{\dot r}}{tg\alpha} = 0$$

$$\ddot r - r\dot\theta^2 + \frac{\ddot r}{tg^2\alpha} + \frac{g}{tg\alpha} = 0$$

$$\ddot r - r\dot\theta^2 + \ddot r\frac{\cos^2\alpha}{\sin^2\alpha} + g\frac{\cos\alpha}{\sin\alpha} = 0$$

$$\frac{\ddot r}{\sin^2\alpha} + g\frac{\cos\alpha}{\sin\alpha} - r\dot\theta^2 = 0$$

$$m\ddot r\frac{\cos\alpha}{\sin^2\alpha} + mg\frac{\cos^2\alpha}{\sin\alpha} = mr\dot\theta^2\cos\alpha \overset{(3)}{=} R - mg\sin\alpha$$

$$m\ddot r = R\frac{\sin^2\alpha}{\cos\alpha} - mg\frac{\sin^2\alpha}{\cancel{\sin\alpha}\cos\alpha} \Rightarrow \text{è uguale alla I eq. di (1)}$$

Combinando queste eqn. con (3)

$$m\ddot r - mr\dot\theta^2 = R\frac{\sin^2\alpha}{\cos\alpha} - \cancel{mg\frac{\sin\alpha}{\cos\alpha}} - \frac{R}{\cos\alpha} + \cancel{mg\frac{\sin\alpha}{\cos\alpha}}$$

$$= R\frac{-(1-\sin^2\alpha)}{\cos\alpha} = -R\cos\alpha$$

$\Downarrow$

è la II eqn. di (1) !

Quanto deve essere $\dot\theta$ per avere un moto circolare?

Moto circ. $\Rightarrow$ $\dot r = \ddot r = 0$ $\Rightarrow$ (1) diventa

$$\begin{cases} -m r \dot\theta^2 = -R\cos\alpha \\ mg = R\sin\alpha \end{cases} \Rightarrow \quad \cancel{m} r \dot\theta^2 = \cancel{m} g \frac{\cos\alpha}{\sin\alpha}$$

$$\dot\theta = \sqrt{\frac{g}{r}\frac{\cos\alpha}{\sin\alpha}}$$

Trovare r_{min} e r_{max} (si potrà avere $r_{min} = 0$?)

$$E = \frac{1}{2} m \dot r^2 \left(1 + \frac{1}{tg^2\alpha}\right) + \frac{1}{2} m r^2 \dot\theta^2 + mg \frac{r}{tg\alpha}$$

$L_{o_z} = m r^2 \dot\theta$ ed è costante $\Rightarrow$

$$E = \frac{1}{2} m \frac{\dot r^2}{\sin^2\alpha} + \frac{1}{2} \cancel{m} \cancel{r^2} \frac{L_{o_z}^2}{m^{\cancel{2}} r^{\cancel{4}\,2}} + mg\, r \frac{\cos\alpha}{\sin\alpha}$$

$$= \frac{1}{2} m \left(\frac{\dot r}{\sin\alpha}\right)^2 + \frac{L_{o_z}^2}{2 m r^2} + mg\, r \frac{\cos\alpha}{\sin\alpha} = \text{cost.}$$

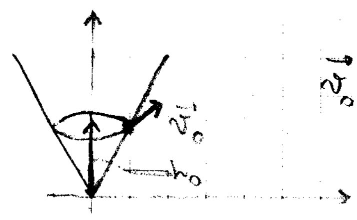
Se $r = 0$ $\Rightarrow$ $E = \infty$ $\Rightarrow$ non potrò mai avere $r = 0$

r_{max} e r_{min} $\Rightarrow$ $\dot r = 0$ $\Rightarrow$

$$E = \frac{L_{o_z}^2}{2 m r^2} + mgr \frac{\cos\alpha}{\sin\alpha}$$

$$2m^2 g \frac{\cos\alpha}{\sin\alpha} r^3 - 2 m r^2 E + L_{o_z}^2 = 0$$

Supponiamo che a $t = 0$



$\vec v_0 = v_0 \hat e_\theta$

$r_0 = h_0\, tg\alpha$

$$\Rightarrow \quad E = \frac{1}{2} m v_0^2 + m g h_0$$

$$L_{o_z} = m h_0\, tg\alpha\, v_0$$

$$2\,m^2 g\, r^3 \frac{\cos\alpha}{\sin\alpha} - 2 m r^2 \left(\frac{1}{2} m v_0^2 + m g \frac{r_0}{\sin\alpha}\cos\alpha \right)$$

$$+ \; m^2 r_0^2 \frac{\cancel{\cos^2\alpha}}{\cancel{\sin^2\alpha}} \cancel{\left(\frac{\sin\alpha}{\cos\alpha}\right)^2} v_0^2 = 0$$

$$2 m^2 g \frac{\cos\alpha}{\sin\alpha} r^3 - m^2 v_0^2 (r^2 - r_0^2) - 2 m^2 g\, r^2 r_0 \frac{\cos\alpha}{\sin\alpha} = 0$$

$$2 \cancel{m^2} g \frac{\cos\alpha}{\sin\alpha} r^2 \cancel{(r - r_0)} - \cancel{m^2} v_0^2 \cancel{(r - r_0)} (r + r_0) = 0$$

$$2 g \frac{\cos\alpha}{\sin\alpha} r^2 - v_0^2 r - v_0^2 r_0 = 0$$

$$r = \frac{+v_0^2 \pm \sqrt{v_0^4 + 8 g v_0^2 \frac{\cos\alpha}{\sin\alpha} r_0}}{4 g \frac{\cos\alpha}{\sin\alpha}}$$

$$\Rightarrow \begin{cases} r_{min} = \dfrac{v_0^2 \sin\alpha}{4 g \cos\alpha} \left[1 + \sqrt{1 + \dfrac{8 g r_0 \cos\alpha}{v_0^2 \sin\alpha}} \right] \\[2ex] r_{max} = r_0 \end{cases}$$