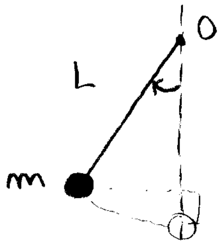
Esercizio sul pendolo matematico e pendolo fisico

Pendolo matematico



Eq. del moto con
i) cons. energia
ii) mom angolare

i) $E = \frac{1}{2} m (L\dot{\theta})^2 + mg L (1-\cos\theta)$

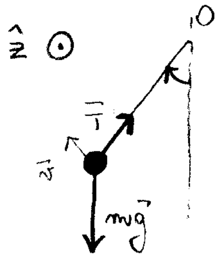
$\frac{dE}{dt} = 0 \Rightarrow \cancel{\frac{1}{2}} \cancel{m} L^2 \cancel{2} \cancel{\dot{\theta}} \ddot{\theta} + \cancel{m} g \cancel{L} \sin\theta \cancel{\dot{\theta}} = 0$

$$\ddot{\theta} + \frac{g}{L} \sin\theta = 0$$

Piccole oscillazioni $\Rightarrow \theta \approx \sin\theta \Rightarrow \ddot{\theta} + \frac{g}{L}\theta = 0$

$$\Rightarrow T = 2\pi \sqrt{\frac{L}{g}}$$

ii) Momento delle forze rispetto ad O



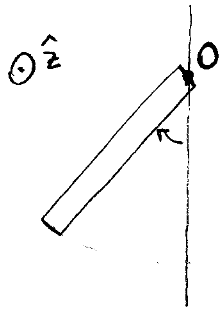
$\vec{M}_o = mg L \sin\theta \ \hat{z}$

$\vec{L}_o = m L^2 \dot{\theta} (-\hat{z})$

$\Rightarrow \cancel{m} L^2 \ddot{\theta} = -\cancel{m} g \cancel{L} \sin\theta$

$$\ddot{\theta} + \frac{g}{L} \sin\theta = 0$$

Pendolo fisico



Barra lunga 2L, densità omogenea λ

Massa tot. m

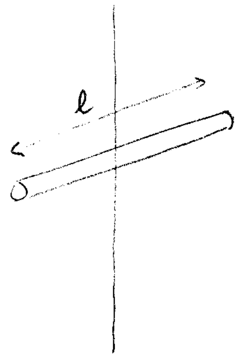
i) $E = \frac{1}{2} I_o \dot{\theta}^2 + mgL(1-\cos\theta)$

Forza costante si applica al baricentro !!

$I_o = I_{CM} + mL^2$

$\Rightarrow \quad I_o \cancel{\dot{\theta}} \ddot{\theta} + mgL \sin\theta \cancel{\dot{\theta}} = 0$

$$\ddot{\theta} + \frac{mgL}{I_o} \sin\theta = 0$$



$I = \frac{1}{12} ml^2$

$I_{CM} = \frac{1}{\cancel{12}\,3} m \cancel{4} L^2 = \frac{1}{3} mL^2$

$\Rightarrow \quad I_o = \frac{4}{3} mL^2$

$\Rightarrow \quad \ddot{\theta} + \frac{\cancel{m}g\cancel{L}}{\frac{4}{3}\cancel{m}L^{\cancel{2}}} \sin\theta = 0$

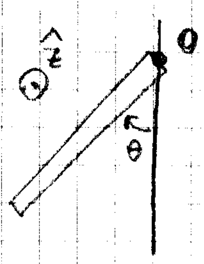
$\ddot{\theta} + \frac{3}{4}\frac{g}{L} \sin\theta = 0 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{4}{3}\frac{L}{g}} > T_{\text{pendolo mat.}}$

ii) $mgL\sin\theta\, \hat{z} = \vec{M}_o$

$\vec{L}_o = I_o \dot{\theta} (-\hat{z}) \quad \Rightarrow \quad I_o \ddot{\theta} = -mgL\sin\theta$

$$\Rightarrow \quad \ddot{\theta} + \frac{mgL}{I_o} \sin\theta = 0$$

Pendolo fisico

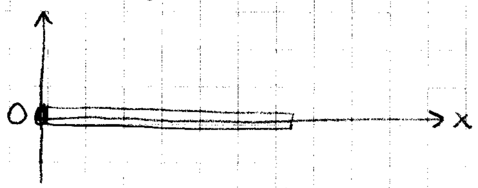


lunghezza $2L$, densità omogenea λ

massa totale m

$$E = \frac{1}{2} I_0 \dot{\theta}^2 + mgL(1-\cos\theta)$$

I_0 ?



$m = 2L\lambda$

$$I_0 = \int dm\, r^2 = \int_0^{2L} x^2 \overbrace{\lambda dx}^{dm} = \lambda \left.\frac{x^3}{3}\right|_0^{2L} = 8\frac{L^3}{3}\lambda = \frac{4}{3} L^2 m$$

$\Rightarrow$ in generale I_0 di una sbarra lunga l è $\frac{1}{3} m l^2$.

$$\Rightarrow \quad E = \cancel{\frac{1}{2}}\, \frac{\cancel{4}^{\,2}}{3} m L^2 \dot{\theta}^2 + mgL(1-\cos\theta)$$

$$\frac{dE}{dt} = 0 = \frac{2}{3} m L^{\cancel{2}}\, 2 \cancel{\dot{\theta}}\, \ddot{\theta} + mg \cancel{L} \cancel{\dot{\theta}} \sin\theta$$

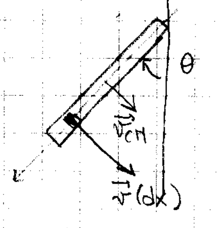
$$\ddot{\theta} + \frac{3}{4}\frac{g}{L}\sin\theta = 0 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{4}{3}\frac{L}{g}} > T_{\text{pendolo mat.}}$$

Perché $T_{\text{pendolo fisico}} > T_{\text{pendolo mat.}}$ $\Leftrightarrow$ perché $E_{\text{pendolo fisico}} \neq E_{\text{pend. mat.}}$?

Vedo il moto come [traslazione del c.m. + rotazione intorno al c.m.]

$$\Rightarrow \quad E = \frac{1}{2} m v_{CM}^2 + \frac{1}{2}\int dm\, v'^2(dx) + mgL(1-\cos\theta)$$

$v_{CM} = L\dot{\theta}$



$v(dx) = x\dot{\theta}$ $\Rightarrow$

$v'(dx) = (x-L)\dot{\theta}$

$$E = \frac{1}{2} m L^2 \dot{\theta}^2 + \frac{1}{2} \int_0^{2L} \lambda dx \, (x-L)^2 \dot{\theta}^2 + mgL(1-\cos\theta)$$

$$= \frac{1}{2} m L^2 \dot{\theta}^2 + \frac{1}{2} \lambda \dot{\theta}^2 \int_{-L}^{L} dy \, y^2 + mgL(1-\cos\theta)$$

$$= \frac{1}{2} m L^2 \dot{\theta}^2 + \frac{1}{2} \lambda \dot{\theta}^2 \frac{2L^3}{3} + mgL(1-\cos\theta)$$

$$= \frac{1}{2} m L^2 \dot{\theta}^2 + \frac{1}{2} m \frac{1}{3} L^2 \dot{\theta}^2 + mgL(1-\cos\theta)$$

$$= \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I_{CM} \omega^2 + mgL(1-\cos\theta)$$

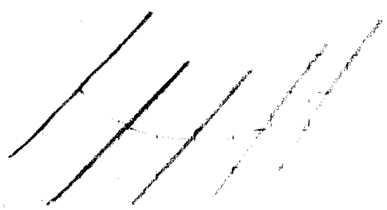
$$\Rightarrow \begin{cases} I_{CM} = \frac{1}{3} m L^2 \\ \omega = \dot{\theta} \end{cases}$$

$\Rightarrow$ I_{CM} di una sbarretta di lunghezza l è $\frac{1}{12} m l^2$

$$E_{\substack{\text{pendolo}\\\text{fisico}}} = E_{\substack{\text{pendolo}\\\text{mat}}} + \underbrace{\frac{1}{2} I_{CM} \dot{\theta}^2}$$

$\Rightarrow$ ho un pezzo in più, che corrisponde alla rotazione della sbarra intorno al c.m.

Se avevo

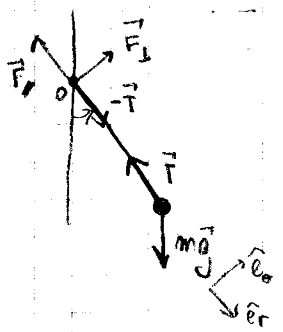


Sbarra resta //

allora $T = T_{\substack{\text{pendolo}\\\text{mat.}}}$

Studiamo la forza di contatto in O:

Pendolo matematico:



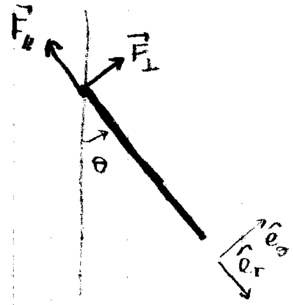
$$\vec{v}_{CM} = +L\dot{\theta}\,\hat{e}_\theta$$

$$m\vec{a}_{CM} = m\vec{g} + \vec{T} \quad ; \quad \vec{a}_{CM} = -L\dot{\theta}^2\,\hat{e}_r + L\ddot{\theta}\,\hat{e}_\theta$$

$$\begin{cases} -L\dot{\theta}^2 m = -T + mg\cos\theta \\ L\ddot{\theta}\, m = -mg\sin\theta \end{cases}$$

$$m\vec{a}_O = 0 = \vec{F}_\parallel + \vec{F}_\perp - \vec{T} \Rightarrow \begin{cases} F_\perp = 0 \\ F_\parallel = T \end{cases}$$

Pendolo fisico:



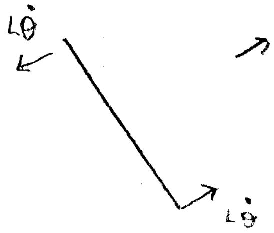
$$m\vec{a}_{CM} = m\vec{g} + \vec{F}_\parallel + \vec{F}_\perp$$

$$\begin{cases} -mL\dot{\theta}^2 = -F_\parallel + mg\cos\theta \\ mL\ddot{\theta} = +F_\perp - mg\sin\theta \end{cases}$$

$$\ddot{\theta} = -\frac{3}{4}\frac{g}{L}\sin\theta \Rightarrow \quad F_\perp = +\frac{3}{4}mg\sin\theta + mg\sin\theta$$

$$= \frac{1}{4}mg\sin\theta$$

SR CM



$\nearrow \vec{v}_{CM} = L\dot{\theta}\,\hat{e}_\theta$

e in questa maniera nel SRA la sbarretta resta allineata

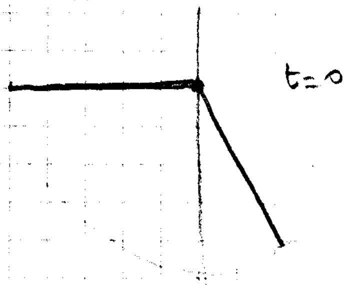
$$\Rightarrow \vec{L}_{CM} = \int dm\, \vec{r}\times\vec{v} = I_{CM}\,\dot{\theta}\,\hat{z} = \frac{1}{3}mL^2\dot{\theta}\,\hat{z}$$

$$\dot{\vec{L}}_{CM} = \vec{M}_{CM} \Rightarrow \quad \frac{1}{3}mL^{\cancel{2}}\ddot{\theta} = -\cancel{L}F_\perp \Rightarrow$$

$$F_\perp = \cancel{+}\frac{\cancel{1}}{\cancel{3}}m\cancel{L}\left(\cancel{+}\frac{g}{\cancel{L}}\right)\frac{\cancel{3}}{4}\sin\theta = \frac{1}{4}mg\sin\theta$$

$\Rightarrow$ occorre $F_\perp$ per dare un momento e far sì che la sbarretta resti allineata.

supponiamo

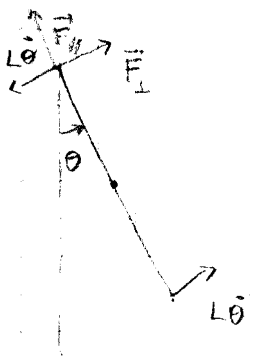


$$E(t=0) = mgL$$

$$\cancel{mgL} = \frac{2}{3} mL^2 \dot{\theta}^2 + mgL(\cancel{1} - \cos\theta)$$

$$\Rightarrow \quad mgL\cos\theta = \frac{2}{3} mL^2 \dot{\theta}^2 \quad (*)$$

Calcoliamo il lavoro fatto da $\vec{F}$ nel SRCM

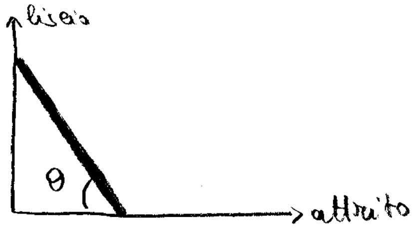


$$\Rightarrow \quad d\mathcal{L} = \vec{F}_{\perp} \cdot d\vec{v}\, dt$$
$$= -F_{\perp} L\dot{\theta}\, dt$$
$$= -\frac{1}{4} mg \sin\theta\, L\dot{\theta}\, dt$$
$$= -\frac{1}{4} mgL \sin\theta\, d\theta$$

$$\Rightarrow \quad \mathcal{L} = \int_{-\frac{\pi}{2}}^{\theta} \left(-\frac{1}{4}\right) mgL \sin\theta\, d\theta = \frac{1}{4} mgL\cos\theta \Big|_{-\pi/2}^{\theta}$$

$$= \frac{1}{4} mLg\cos\theta \underset{(*)}{=} \frac{1}{4}\,\frac{2}{3} mL^2\dot{\theta}^2 = \frac{1}{6} mL^2\dot{\theta}^2$$
$$= \frac{1}{2} I_{CM}\dot{\theta}^2$$

Esercizio della scala

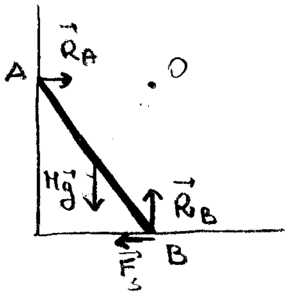


L, M

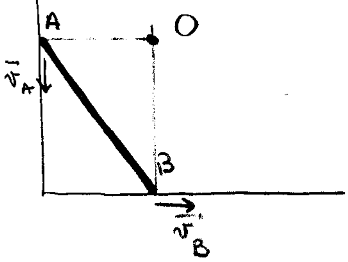
μ_s

1) Trovare θ_{min} prima di scivolare

Forze



; Moto



$\Rightarrow$ centro istantaneo di rotazione è O.

se ho equilibrio $\sum \vec{F} = 0$ forze

$\sum \vec{M} = 0$ mom delle forze

Calcolo $\vec{M}$ rispetto ad O $\Rightarrow$ rimane solo momento del peso e della forza di attrito:

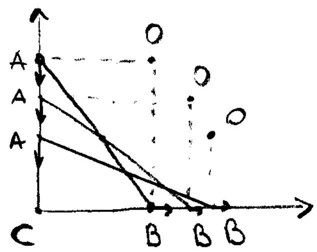
$$\frac{L}{2} Mg \cos\theta_{min} - F_s L \sin\theta_{min} = 0$$

$$\begin{cases} R_A = F_s \\ R_B = Mg \end{cases} ; \text{ so che } F_s = R_B \mu_s \Rightarrow F_s = \mu_s Mg$$

$$\Rightarrow \frac{\cancel{L}}{2} \cancel{Mg} \cos\theta_{min} - \cancel{Mg} \mu_s \cancel{L} \sin\theta_{min} = 0$$

$$\boxed{\operatorname{tg}\theta_{min} = \frac{1}{2\mu_s}}$$

2) Tolgo l'attrito. Descriviamo il moto:



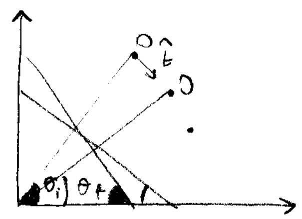
- Rapporto v_A e v_B:

$$v_A = L\cos\theta\,\dot\theta$$
$$v_B = L\sin\theta\,\dot\theta \quad \Rightarrow \quad \boxed{\frac{v_A}{v_B} = \frac{1}{tg\theta}}$$

- Che moto fa il centro istantaneo di rotazione O?

Moto circolare intorno a C: $\overline{CO}$ è sempre costante $= L$

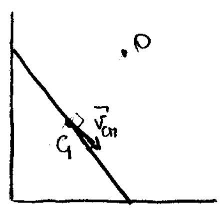
Inoltre $\dot\theta_O = \dot\theta$:



$$d\theta_O = \theta_f - \theta_i = d\theta$$

$$\Rightarrow \boxed{\vec v_O = L\dot\theta\,\hat t}$$

Nota che se O è il centro istantaneo di rotazione, allora anche CM ruota intorno ad O $\Rightarrow \vec v_{CM} \perp \overline{GO}$



Troviamo $\ddot\theta$: $\quad I_O(-\ddot\theta) = M_O$

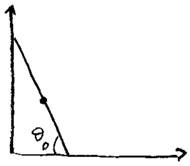
0 perché $\vec v_O // \vec P$

$$M_O = Mg\frac{L}{2}\cos\theta + \cancel{|\vec v_O \times \vec P|}$$

$$\Rightarrow I_O\ddot\theta = -Mg\frac{L}{2}\cos\theta \qquad I_O = I_{CM} + M\frac{L^2}{4} = \frac{1}{12}ML^2 + \frac{1}{4}ML^2$$
$$= \frac{1}{3}ML^2$$

$$\Rightarrow \boxed{\ddot\theta = -\frac{\cancel{M}g\frac{\cancel{L}}{2}\cos\theta}{\frac{1}{3}\cancel{M}L^2} = -\frac{3}{2}\frac{g}{L}\cos\theta}$$

Troviamo $\dot{\vartheta}$:



$t = 0$

$$E_{in} = Mg \frac{L}{2} \sin\vartheta_0$$

$$= E = \frac{1}{2} I_0 \dot{\vartheta}^2 + Mg \frac{L}{2} \sin\vartheta$$

$$\Rightarrow \boxed{\dot{\vartheta} = \sqrt{\frac{2E_0 - MgL\sin\vartheta}{I_0}}}.$$