Introduction to Conformal Field Theories

Alessandro Vichi[a]

[a] *Dipartimento di Fisica dell'Università di Pisa, Largo Pontecorvo 3, I-56127 Pisa, Italy*

Abstract

Lecture notes for the third module of the master course at University of Pisa: Non Perturbative Aspects of QFT - academic year 2019/2020

Contents

References

- DSD: TASI Lectures on the Conformal Bootstrap, arXiv:1602.07982
- SR: EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions, arXiv:1601.05000
- DP, SR, AV: The Conformal Bootstrap: Theory, Numerical Techniques, and Applications arXiv:1805.04405

1 Introduction

1.1 Motivations

The first part of the course introduced the Renormalization Group (RG) flow as a trajectory in the space of theories. The end point of these trajectories are called *fixed points* and play a special role on in our understanding of quantum physics. We can indeed think about a Quantum Field Theory (QFT) as the interpolation between different fixed points, which act like signpost, or milestones.

Understanding and classifying these fixed points represent a fundamental topic of modern theoretical physics. Why is so? First, they are interesting per se, as they describe the long distance behaviour of physical systems. When only interested in the long distance behaviour of the theory it makes sense to integrate our the high momentum modes of our fields[1] and work with an effective action where only the important degrees of freedom appear. In doing so we get the so called Wilson effective action. As we have seen, the effect of integrating out shell of momenta is to redefine the coupling constant appearing in our Lagrangian. More specifically, coupling constants are suppressed or enhanced according to the dimension of the interaction they parametrize. We can then distinguish

- relevant operators $[\mathcal{O}] < 4$: they become important in the IR
- marginal operators $[\mathcal{O}] = 4$: they are equally important at all scales
- irrelevant operators $[\mathcal{O}] > 4$: they get suppressed in the IR

The above classification makes it clear that, given a set of fundamental fields and a given global symmetry, there is only a finite set of interactions that can survive in the IR limit. All the details about the initial theory are washed out along the RG flow. This observation leads to the concept of universality: physical systems that only differ by a choice of irrelevant operators are described by the same fixed point at long distances. This is the reason why ferromagnets at the critical temperature, or boiling water at the tricritical point the pressure-temperature plane, are described at long distances by the same fixed point, the

[1] These modes are not produced as external states in our scattering experiments and their contribution is suppressed inside loops.

three dimensional Ising model. This is also the reason why MonteCarlo simulations of lattice of spins agree so well with quantum field theory computations in terms of a single scalar field: because the UV details do not matter.

In fact the concept of universality has stronger implications: under certain circumstances, even systems with different values of relevant coupling can be described by the same fixed point. Let us see a simple and intriguing example: the Gross-Neveu-Yukawa Model in three dimension. It is described by a simple lagrangian:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 + i\overline{\psi}\,\partial\!\!\!/\psi + g_1 \phi \overline{\psi}\psi + \frac{g_2}{4!}\phi^4 \tag{1.1}$$

where ϕ is a real field and ψ is a Majorana fermion. The β functions for the above model computed in $4 - \varepsilon$ are

$$\begin{aligned} \beta_{g_2} &= -\varepsilon g_2 + \frac{1}{(4\pi)^2}\left(3g_2 + 8g_1^2 g_2 - 48 g_1^4\right) \\ \beta_{g_1} &= -\frac{\varepsilon}{2} g_1 + \frac{5}{(4\pi)^2} g_1^3 \end{aligned} \tag{1.2}$$

Looking at the common zeros, we obtain the picture shown in Fig. 1. The RG flow of the

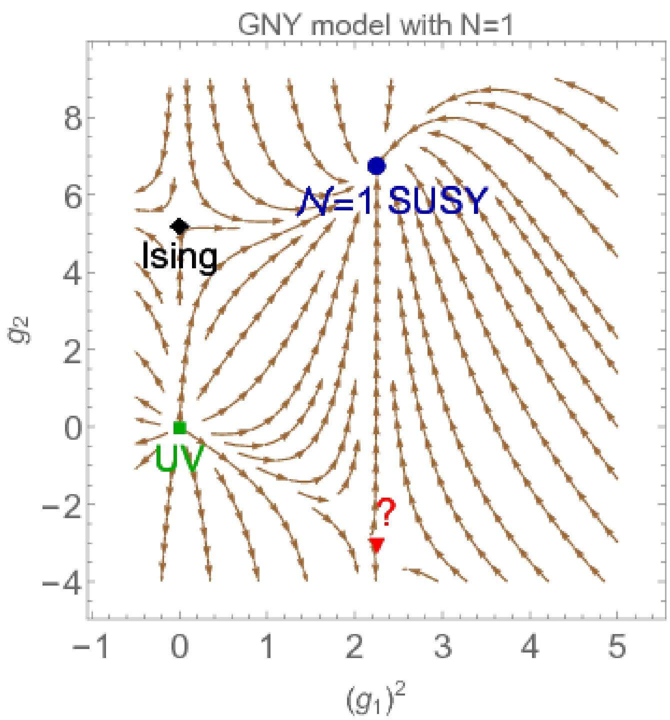

Figure 1: Plot of RG flows on the GNY model in three dimension. The plot has been obtained interpolating a $4 - \varepsilon$ expansion with a $2 + \varepsilon$ using a Padé approximation.

GNY model is such that if we start anywhere in the parameter space (except on the $g_1 = 0$ line), the IR behaviour will be described by the same theory. Not only, this theory has an amazing characteristic: it posses a symmetry exchanging fermions and bosons, namely a supersymmetry. This is an example of an *emergent symmetry* in the IR.

Exercise 1.1. *Consider the following lagrangian with two scalar fields*

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4) - \frac{2\rho}{4!}\phi_1^2\phi_2^2 \tag{1.3}$$

Notice that for $\rho = \lambda$, this lagrangian has an $O(2)$ invariance rotating the two fields into one another.

- *Working in four dimensions, find the β functions for the two coupling constants to leading order.*
- *Write the renormalization group equation for the ration ρ/λ. Show that if $\rho/\lambda < 3$ at a given renormalization point M, this ratio flows towards $\rho = \lambda$ at large scales.*
- *Write the β functions for the couplings in $4 - \epsilon$ dimensions. Show that there are non-trivial fixed points of the renormalization group flow at $\rho/\lambda = 0, 1, 3$. Which is the most stable? Sketch the pattern of coupling constant flows.*

Another important application of fixed point is for the theory of critical phenomena: second order phase transitions are described by scale invariant fixed points.

Finally, the knowledge of a fixed point allows to describe its neighbourhood. You have implicitly been doing this whenever you did the usual perturbative expansion. In that case the starting point was still a fixed point, albeit a trivial one: a free theory. The complete knowledge of all correlation functions allows to compute observables even in interacting theory, as a controlled expansion in terms of the perturbation. More specifically, given an action

$$\mathcal{S} = \mathcal{S}^{\text{f.p.}} + g \int d^d x \mathcal{O}(x) \tag{1.4}$$

one defines a Path Integral in the perturbed theory in terms of the original one:

$$\int \mathscr{D}[\phi] e^{\frac{i}{\hbar}\mathcal{S}[\phi]} = \prod_n \frac{(ig/\hbar)^n}{n!} \int dx_1^d \dots \int dx_n^d \int \mathscr{D}[\phi] e^{\frac{i}{\hbar}\mathcal{S}^{\text{f.p.}}[\phi]} \mathcal{O}(x_1)\dots\mathcal{O}(x_n) \tag{1.5}$$

This implies that any correlation function can be computed in terms od known correlation functions, ex:

$$\langle \mathcal{O}_1(y_1)\mathcal{O}_2(y_2)\rangle = \langle \mathcal{O}_1\mathcal{O}_2\rangle_{\text{f.p.}} + \frac{ig}{\hbar}\int dx \langle \mathcal{O}_1(y_1)\mathcal{O}_2(y_2)\mathcal{O}(x)\rangle_{\text{f.p.}} + \dots \tag{1.6}$$

When the fixed point is the free theory, the above expression gives rise to the usual Feynman diagrams expansion. More generically, the above expansion is called *conformal perturbation theory.*

For all these reasons we dedicate this module to the understanding of the properties of scale invariant fixed points: how to describe them, how to classify them and how to compute observables. Since this course focuses on non perturbative aspects of QFT, the idea is to develop a set of techniques that can be used whenever the usual perturbative methods fail. As an example, in Fig. .1 the value of the couplings are such that the usual perturbation theory does not apply. As an example of how hard these computations can get, let us consider the computation of the critical exponents for the three dimensional Ising model:

$$\begin{aligned} \text{Correlation Length}: &\quad \xi \sim (T-T_c)^{-\nu} \\ \text{Heat Capacity}: &\quad C \sim (T-T_c)^{2-3\nu} \\ \text{Magnetic Susceptibility}: &\quad \chi \sim (T-T_c)^{(2-\eta)/\nu} \end{aligned} \tag{1.7}$$

The prediction in Fig. 2 have been obtained by computing the β-function to 7 loops and Borel-resumming the perturbation theory. And yet they are vey unprecise compared to numerical methods. In oder to understand the theory in the IR we need to be able to independently understand the new fixed points.

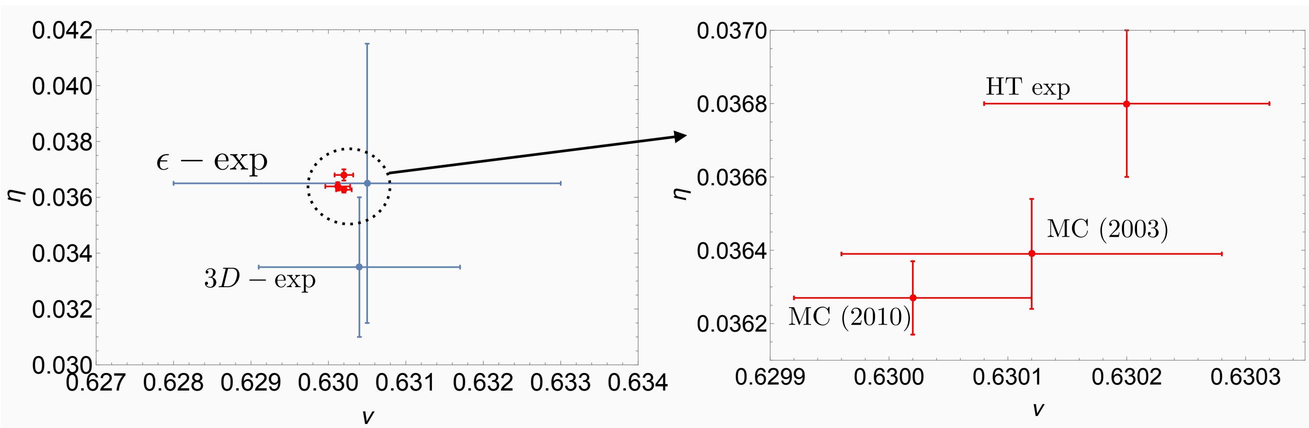

Figure 2: Predictions for the critical exponents η, ν defined in Eq. (1.7).

[end of lecture 1]

1.2 Local Theories and the Stress Tensor

For the purposes of these lectures we would like to restrict ourself theories that posses a concept of locality. In order to understand what we mean by local, we give a precise definition: *the theory is local if it has a local stress tensor operator with the following properties:*

- Conserved away from other operators $\partial_\mu T^{\mu\nu} = 0$
- Symmetric
- Generate translations

In order to understand what those requirement mean, let us review how symmetries play a role in the Path Integral approach to QFT.

1.3 Ward identities and topological operators

Let us start from the Path Integral $\mathcal{Z}[J]$ and perform the change of variable in the integrated field Ψ_i:

$$\Phi_i \longrightarrow f^a(x)\delta_a\Phi_i(x)\,. \tag{1.8}$$

Then the partition functions changes according to[2]:

$$\mathcal{Z}[J] = \int \mathscr{D}[\Phi] e^{i\mathcal{S}[\Phi_i + f^a\delta_a\Phi_i] + \int d^dx J^i(\Psi_i + f^a\delta_a\Phi_i)} \tag{1.9}$$

Taking the linear terms in f^a and equating the result to zero (recall this is just a change of variable of integration) one gets

$$\int \mathscr{D}[\Phi] \int d^dx \left(\underbrace{-\frac{\partial\mathcal{S}}{\partial\Phi_k}\delta_a\Phi_k}_{\partial_\mu J_a^{N\mu}} + iJ^i\delta_a\Phi_i \right) e^{i\mathcal{S}[\Phi_i] + \int d^dx' J^i\Psi_i} \tag{1.10}$$

where J^N denotes the classical Noether current associated to the symmetry transformation realized on the fields. If we now further derive the above expression with respect to (w.r.t.) the source $J^i(y)$ we get:

$$\partial_\nu\langle 0|T\left\{J_a^{N\mu}(x)\Phi_i(y)\right\}|0\rangle = -i\delta^{(d)}(x-y)\langle 0|\delta_a\Phi_i(x)|0\rangle \tag{1.11}$$

The generalization to multiple insertions is straightforward:

$$\partial_\nu\langle 0|T\left\{J_a^{N\mu}(x)\prod_{i=1}^n\Phi_i(y_i)\right\}|0\rangle = -i\sum_{i=1}^n\delta^{(d)}(x-y_i)\langle 0|T\left\{\delta_a\Phi_i(x)\prod_{j\neq i}\Phi_j(y_j)\right\}|0\rangle \tag{1.12}$$

We can now explore the consequences of the above expression by integrating over the spacial coordinates $d^{d-1}x$, keeping the time coordinate fixed. Splitting the divergence of the Noether current in space and time component, we get a term with is the space integral of a total derivative and a second terms. Dropping boundary terms one is left with:

$$\int d^{d-1}x\frac{\partial}{\partial x^0} < 0|T\left\{J_a^{N0}(x)\Phi_i(y)\right\}|0\rangle = -i\delta(x^0-y^0)\langle 0|\delta_a\Phi_i(x)|0\rangle \tag{1.13}$$

Recalling the definition of the charge $Q_a = \int d^{d-1}xJ_a^{N0}(x)$ and the definition of time-ordered product of operators

$$\begin{aligned}\langle 0|T\left\{A(x^0,x^i)B(y^0,y^i)\right\}|0\rangle \;\equiv\; & \theta(x^0-y^0)\langle 0|A(x^0,x^i)B(y^0,y^i)|0\rangle \\ & +\theta(y^0-x^0)\langle 0|B(y^0,y^i)A(x^0,x^i)|0\rangle \end{aligned} \tag{1.14}$$

[2]We suppress the dependence on spacetime coordinate whenever is not needed.

we obtain a piece proportional to $\delta(x^0 - y^0)$ and a regular piece. Let us procede under the assumption that operators are inserted at distinct times. Then this last terms always vanish and one get:

$$\{\left[Q_a(x^0), \Phi_a(y)\right]\rangle = -i\langle\delta_a\Phi_i(y)\rangle\,, \tag{1.15}$$

The meaning of the above equation is quite profound: the charge defined as the space integral of the Noether current is a *generator* of the symmetry transformation in Eq. (3.6), in the sense that inside any correlation function its commutator acts as a generator.

So far we have been implicitly assumed that the theory is quantized along the time direction, which means it make sense to foliate the space-time with equal time slices. This assumption entered in the definition of time-ordering. We will see however that this is not always the most convenient choice. Let us therefore consider the consequences of Eq. (1.10) and its J^i derivatives when integrating on a more generic region B with boundary $\Sigma = \partial B$.

$$\int \mathscr{D}[\Phi]e^{i\mathcal{S}[\Phi_i]}Q_a(\Sigma)\prod_{i=1}^{n}\Phi_i(y_i) = -i\sum_{y_i\in B}\int \mathscr{D}[\Phi]e^{i\mathcal{S}[\Phi_i]}\delta_a\Phi_i(y_i)\prod_{j\neq i}\Phi_j(y_j) \tag{1.16}$$

Notice that the above expression does not make any reference to a particular quantization. Here $Q_a(\Sigma)$ is the integral on the boundary of the region B of the normal component to the surface of the current $J_a^{N\mu}$. An important consequence of the above is that if the region B doesn't contain any point x_i, the lhs is also zero.
This means that the operator $Q_a(\Sigma)$ is a topological operator: if we change the shape of Σ continuously without crossing any point in the process, then the lhs is invariant. To see this, it is sufficient to consider the difference of Eq. (1.17) and a a similar expression where we integrate on Σ'. If Σ and Σ' can be deformed one into the other without crossing any operator insertion, then the difference is zero. What happens instead when we do cross a point, let us say $i = 1$ for concreteness?. Then the rhs only contains a sum over points such that $y_i \in B$ and $y_i \notin B'$:

$$\int \mathscr{D}[\Phi]e^{i\mathcal{S}[\Phi_i]}\left(Q_a(\Sigma) - Q_a(\Sigma')\right)\prod_{i=1}^{n}\Phi_i(y_i) = -i\int \mathscr{D}[\Phi]e^{i\mathcal{S}[\Phi_i]}\delta_a\Phi_1(y_1)\prod_{j\neq 1}\Phi_j(y_j) \tag{1.17}$$

Deforming continuously Σ and Σ' the lhs can be rewritten as the integral over a $(d-1)$-dimensional sphere of arbitrary small radius surrounding the point x_1. In doing this we have shown that the symmetry transformation under a symmetry of a field are a local property: even if the charge is defined as the integral over an extended surface, its popological properties make it such that it only depends on the field it acts on.

1.4 Scale invariance or more?

RG tranformation

$$x \to x' = (1+\lambda)x \Leftrightarrow g_{\mu\nu} \to (1+\lambda)^2 g_{\mu\nu} \tag{1.18}$$

Under such an infinitesimal rescaling $\lambda \ll 1$,the action changes according to

$$\delta S = \int d^d x \sqrt{g} T_{\mu\nu} \delta g_{\mu\nu} \propto \int d^d x T^{\mu}_{\ \mu} \tag{1.19}$$

The tensor appearing in the above expression is called Energy momentum tensor, or Stress Tensor. It can be defined in several ways: it is the Noether current associated to Poicaré invariance

$$T_{\mu}^{\ \nu} = \sum_i \frac{\partial \mathscr{L}}{\partial(\partial_\nu \Phi_i)} \partial_\mu \Phi_i - \delta_\mu^{\ \nu} \mathscr{L} \,, \tag{1.20}$$

but it can also be defined as the response of the systems under perturbation of the background geometry:

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\partial \mathcal{S}}{\partial g_{\mu\nu}} \,. \tag{1.21}$$

Exercise 1.2. *Show that the two definitions agree*

For the theory to be scale invariant the integral in Eq. (1.19) must vanish. Generically this implies that

$$T^{\mu}_{\ \mu} = \partial_\mu K^\mu \tag{1.22}$$

If one insists that $[T^{\mu}_{\ \mu}] = d$[3], then K_μ would have $[K_\mu] = d - 1$. Notice that generically all operators in QFT acquire an anomalous dimension. If that is the case, then there would not be candidates for the vector K_μ.

When the condition of traceless-ness is enforced the theory is actually invariant under a larger set of transformation, namely coordinate dependent rescaling.

$$\delta S = \int d^d x \sqrt{g} T_{\mu\nu} \delta g_{\mu\nu} \propto \int d^d x \sqrt{g} T^{\mu}_{\ \mu} \lambda(x) \tag{1.23}$$

These set of transformations are called Weyl transformations. We then conclude that a theory with vanishing $T^{\mu}_{\ \mu}$ is invariant under infinitesimal Weyl transformations. This fact has deep consequences on the physics of a system. Extending infinitesimal transformations to finite one is not automatic, because of the possible presence of anomalies. If it was passible then one could in principle compute observables in any metric that is proportional to the flat one.

Here we will restrict our attention only to a subset of Weyl transformations. An infinitesimal Weyl transformation generically is not just a coordinate transformation: it changes substantially the metric and can deform a flat spacetime to a curved one. There is however a subset of these transformations for which the space remains flat. This is the conformal group.

[3] The trace of the Stress Tensor is the generator of dilatation, and as such its dimension is fixed.

Notice that when we discuss quantum field theory, the effect of a scale transformation is more than merely rescaling the metric. Also the coupling constants of the theory change. This means that there are additional terms that can contribute to the trace of the stress tensor. If a coupling changes as $g \to \lambda g \beta(g)$ then the action contains a term of the form

$$\delta S \supset \lambda \beta(g) \int d^d x \frac{\partial \mathcal{L}}{\partial g} \tag{1.24}$$

Exercise 1.3. *Consider the the Lagrangian of QED*

$$\mathcal{L} = -\frac{1}{4e^2}(F_{\mu\nu})^2 + i\overline{\psi}\not{D}\psi \tag{1.25}$$

- *Show that the stress tensor has the form:*

$$T^{\mu\nu} = -\frac{1}{e^2}F^{\mu\rho}F^{\nu}_{\rho} + \frac{1}{4e^2}g^{\mu\nu}(F_{\rho\sigma})^2 + \frac{i}{2}\overline{\psi}\left(\gamma^\mu D^\nu + \gamma^\nu D^\mu\right)\psi - ig^{\mu\nu}\overline{\psi}\not{D}\psi \tag{1.26}$$

- *Working in $d = 4 - \epsilon$ dimension compute T^{μ}_{μ} and show that it is not zero for $\epsilon \neq 0$.*
- *Compute the amplitude $\langle T^{\mu}_{\mu}(0) A_\rho(k) A_\nu(-k)\rangle$ and show that there is term (finite in the $\epsilon \to 0$ limit) that reproduces the operator identity*

$$T^{\mu}_{\mu} = \frac{\beta(e)}{2e^3}(F_{\mu\nu})^2 \tag{1.27}$$

with $\beta(e) = \frac{e^3}{12\pi^2}$ the β-function of QED.

[end of lecture 2]

2 Conformal Symmetry

In this chapter we present the definition of conformal symmetry and derive its algebra using the explicit differential representation of the generators. Then we discuss the structure of a representation of conformal group and we will describe the restrictions posed by unitarity. Finally we examine how to construct invariants out of four points in space-time.

2.1 Conformal Transformations

Let us consider the metric tensor $g_{\mu\nu}(x)$ of a d-dimensional space-time. The conformal group can be defined as the set of diffeomorphisms that leave the metric unchanged up to a overall scale factor, which in general can be coordinate dependent:

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = \Lambda(x)^2 g_{\mu\nu}(x)\,. \tag{2.1}$$

Another way to phrase this in the flat case is that the Jacobian of the transfomation is an orhogonal metric times a coordinate dependent prefactor.

$$\frac{\partial x^\rho}{\partial x'^\mu} = \Lambda(x) R^{\rho}_{\mu} \tag{2.2}$$

Writing at infinitesimal level $x'^{\mu}(x) = x^{\mu} + \epsilon^{\mu}(x)$ and $\Lambda = 1 - O(\epsilon)$ we can obtain the general constraint[4]

$$\partial_\rho \epsilon_\mu + \partial_\mu \epsilon_\rho = \frac{2}{d} \partial^\sigma \epsilon_\sigma g_{\mu\rho} \, . \tag{2.3}$$

Deriving a second time and permuting the indices (we restrict to constant metric tensor) we get

$$\begin{aligned} \partial_\mu \partial_\nu \epsilon_\rho + \partial_\rho \partial_\nu \epsilon_\mu &= \frac{2}{d} \partial_\nu \partial^\sigma \epsilon_\sigma g_{\mu\rho} \, , \\ \partial_\nu \partial_\mu \epsilon_\rho + \partial_\rho \partial_\mu \epsilon_\nu &= \frac{2}{d} \partial_\mu \partial^\sigma \epsilon_\sigma g_{\nu\rho} \, , \\ \partial_\mu \partial_\rho \epsilon_\nu + \partial_\nu \partial_\rho \epsilon_\mu &= \frac{2}{d} \partial_\rho \partial^\sigma \epsilon_\sigma g_{\mu\nu} \, . \end{aligned} \tag{2.4}$$

Adding the first and the third equation and subtracting the last one we obtain

$$\partial_\rho \partial_\nu \epsilon_\mu = \frac{1}{d} \left(\partial_\nu \partial^\sigma \epsilon_\sigma g_{\mu\rho} - \partial_\mu \partial^\sigma \epsilon_\sigma g_{\nu\rho} + \partial_\rho \partial^\sigma \epsilon_\sigma g_{\mu\nu} \right) \, . \tag{2.5}$$

Finally, contracting the indices, we obtain

$$\Box \epsilon_\mu = \frac{2-d}{d} \partial_\mu (\partial^\sigma \epsilon_\sigma) \, , \tag{2.6}$$

where $\Box$ is defined with the matrix $g_{\mu\nu}$, whatever signature we have chosen. Finally, applying ∂_ν to the above equation and $\Box$ to eq. (2.3) we get:

$$\begin{aligned} \partial_\nu \Box \epsilon_\mu &= \frac{2-d}{d} \partial_\mu \partial_\nu (\partial^\sigma \epsilon_\sigma) \, , \\ \partial_\nu \Box \epsilon_\mu + \partial_\mu \Box \epsilon_\nu &= \frac{2}{d} \Box \partial^\sigma \epsilon_\sigma g_{\mu\nu} \end{aligned} \tag{2.7}$$

Symmetrizing the first equation we can get

$$(2-d) \partial_\mu \partial_\nu (\partial^\sigma \epsilon_\sigma) = g_{\mu\nu} \Box \partial^\sigma \epsilon_\sigma \, , \tag{2.8}$$

and taking the trace of the above equation we finally obtain a second order equation for $f(x) \doteqdot (\partial^\sigma \epsilon_\sigma)$

$$(d-1) \Box f(x) = 0 \, . \tag{2.9}$$

Inserting the above constraint in eq.(2.8) we argue that for $d > 2$ the function $f(x)$ must be at least linear in the coordinates. Hence the general solution is $f(x) = A + B_\mu x^\mu$, which translates in the general expression

$$\epsilon^\mu = c^\mu + a_{\mu\nu} x^\nu + b_{\mu\nu\rho} x^\nu x^\rho \tag{2.10}$$

[4] The rhs can be easily obtained by taking the trace of $\partial_\rho \epsilon_\mu + \partial_\mu \epsilon_\rho = f(x) g_{\mu\rho}$.

Plugging the general solution into eq. (2.5) we observe that the coefficient $b_{\mu\nu\rho}$ can be expressed in terms of only one vector $b_{\sigma\ \mu}^{\ \sigma}$:

$$b_{\mu\nu\rho} = \frac{1}{d}\left(b_{\sigma\ \nu}^{\ \sigma} g_{\mu\rho} + b_{\sigma\ \rho}^{\ \sigma} g_{\mu\nu} - b_{\sigma\ \mu}^{\ \sigma} g_{\nu\rho}\right) \tag{2.11}$$

Finally, using eq. (2.3) we show that the symmetric part of $a_{\mu\nu}$ is proportional to the matrix $g_{\mu\nu}$, while the antisymmetric one is completely unconstrained.
Counting the parameters contained in a general infinitesimal transformation we obtain

$$c_\mu : \quad D\,, \qquad\qquad a_{\mu\nu} : \quad \frac{d(d-1)}{2} + 1\,,, \qquad\qquad b_\mu : \quad D\,. \tag{2.12}$$

for a total of $(d+1)(d+2)/2$ parameters. We can easily recognize the transformations associated to the above parameters:

$$\begin{aligned}
&x'^\mu = x^\mu + c^\mu : \quad \text{translations}\\
&x'^\mu = x^\mu + \lambda x^\mu : \quad \text{dilatations}\\
&x'^\mu = x^\mu + w^\mu_{\ \nu} x^\nu : \quad \text{Lorentz rotations}\\
&x'^\mu = x^\mu + 2(b_\rho x^\rho) x^\mu - x^2 b^\mu : \quad \text{Conformal boosts}
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\varepsilon_\mu(x) &= x_\mu + a_\mu\,, & \Omega^2 &= 1\,,\\
\varepsilon_\mu(x) &= x_\mu + \Lambda_\mu^{\ \nu} x_\nu\,, & \Omega^2 &= 1\,,\\
\varepsilon_\mu(x) &= \lambda x_\mu\,, & \Omega^2 &= \lambda^2\,,\\
\varepsilon_\mu(x) &= \frac{x_\mu - b_\mu x^2}{1 - 2(x\cdot b) + b^2 x^2}\,, & \Omega^2 &= (1 - 2(x\cdot b) + b^2x^2)^2\,,
\end{aligned} \tag{2.14}$$

Exercise 2.1. *Show that in $d = 1$ dimensions any coordinate transformation is a conformal transformation (there are no angles to be preserved).*

Exercise 2.2. *Show that in $d = 2$, given the coordinates (z^0, z^1) with a flat metric on the plane, and the transfomation $z_i \to w_i(z_j)$, the condition in Eq. (2.1) corresponds to*

$$\begin{aligned}
&\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2\\
&\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0
\end{aligned} \tag{2.15}$$

Show that the above conditions are equivalent to the Cauchy-Riemann equations for holomorphic (or anti-holomorphic) functions:

$$\frac{\partial w^1}{\partial z^0} = \pm \frac{\partial w^0}{\partial z^1}\,, \qquad \frac{\partial w^0}{\partial z^0} = \mp \frac{\partial w^1}{\partial z^1} \tag{2.16}$$

Conclude that conformal transformations in $d = 2$ are those for which the function $w(z)$ is holomorphic, where

$$z = z^0 + iz^1\,, \quad \overline{z} = z^0 - iz^1\,, \qquad w = w^0 + iw^1\,, \quad \overline{w} = w^0 - iw^1 \tag{2.17}$$

[end of lecture 3]

2.2 Conformal frames

Given a set of n points, we can make use of conformal transformations to arrange them in convenient configurations. As an example, three points can be set to $y_{1,2,3} = 0, \vec{e}, \infty$, where $\vec{e}$ is any unit vector.[5] In doing so, we haven't exhausted the full symmetry: a residual $SO(d-1)$ remains. This symmetry can be though as the stabilizer group of the points y_i. If there are more than three points the stabilizer group of this configuration is instead $SO(d+2-m)$, where $m = \text{Min}(n, d+2)$.

Exercise 2.3. *Consider the case $d = 2$. Find the modular transformation that send any three point $z_{1,2,3}$ in $w = 0, 1, \infty$. Show that this transformation maps the circle passing through the three points to the real line, and the disk contained inside the circle in the upper(or lower) half plane.*

Notice that when $n > m$, there is no residual symmetry, and there are $(n-m)$ d-dimensional points that are completely unconstrained. The first m points additionally contain $m(m-3)/2$ degrees of freedom. The unconstrained degrees of freedom can be identified with the number of independent conformally invariant cross-ratios that can be built out of n points.

The case we are primarily interested in is $n = 4$. In this case the number of conformal invariants is always 2, except in $d = 1$ which has only one invariant. These invariants can be identified with the cross ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{2.18}$$

Exercise 2.4. *Show that the above expression are invariant under the transformation defined in exercise 2.3.*

To make contact with the configuration above, in Euclidean signature we can introduce the convenient parametrization:

$$\begin{aligned} x_1 &= (0, 0, \vec{0})\,, \\ x_2 &= (0, 1, \vec{0})\,, \\ x_3 &= (\frac{z-\bar{z}}{2i}, \frac{z+\bar{z}}{2}, \vec{0})\,, \\ x_4 &= (0, \infty, \vec{0})\,, \end{aligned} \tag{2.19}$$

where $\vec{0}$ is a $(d-2)$-vector. Here $z, \bar{z}$ are complex conjugate variables. In Minkowski signature they become real and independent. With the above choice we then have the simple relations:

$$u = z\bar{z}\,, \qquad v = (1-z)(1-\bar{z}) \tag{2.20}$$

A given choice of points y_i, as in Eq. (2.19), is called a conformal frame. It can be thought of as a gauge fixing of most of the conformal symmetry. Because any coordinate

[5] Formally ∞ can be though as the $L\vec{e}$, with $L \to \infty$

configuration can be reduced to a given conformal frame and vice-versa, the knowledge of a correlation function in the conformal frame is sufficient to reconstruct it at any generic point through its covariant properties. This means that in order to classify the independent tensor structures or to impose consistency conditions (such as crossing), we can restrict to a conformal frame instead of working at generic coordinate configurations. Also, covariance with respect to the conformal group implies invariance under the stabilizer group.

2.2.1 Preview of correlation functions

As we will see later, in CFTs we are interested in computing correlation functions of local operators. We can already understand how powerful the conformal symmetry can be by comparing the general form of a correlation function of thee scalars in a general QFT with the case of CFT. Consider the correlation function:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle_{QFT} \tag{2.21}$$

Without loose of generality, because of translation invariance, we can set $x_1 = 0$. Then Eq. (2.21) can only depend in x_2 and x_3. Let us work in euclidean space for simplicity. Then using rotations we can always put x_2 along the d-direction and x_3 on a plane in the $(d, d-1)$-directions. We are then left with only three quantities. This of course reflects the original invariants that one could have guessed from the very begginnig based on Poincaré invariance: $(x_1-x_2)^2, (x_2-x_3)^2, (x_1-x_2)\cdot(x_2-x_3)$.

If we repeat the same argument in CFTs: because of the arguments in the previous section, we can always set three points to a fixed location, we are left with no freedom:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle_{CFT} \sim \langle \phi_1(0)\phi_2(\vec{e})\phi_3(\infty)\rangle_{CFT} \sim \text{const} \tag{2.22}$$

We conclude that the three point functions is fixed by kinematic up to a constant factor.

Exercise 2.5. *Repeat the previous argument for the case of two scalars and a vector $\langle V^\mu(x_1)\phi_2(x_2)\phi_3(x_3)\rangle$ in a general QFT and CFT.*

2.3 Algebra of the Conformal Symmetry - I

Starting from the infinitesimal transformations we can define the differential form of the generators acting on functions. Given a coordinate transformation $x \to x' = \xi(x)$ (therefore $x = \xi^{-1}(x')$), we have $f(x) \to f'(x) = f(\xi^{-1}(x))$. The implementation of function can be implemented through differential generators J such that $f'(x) = e^{-iJ}f(x)$. In the case in exam we get:

$$\begin{array}{lll}
\text{Translations}: & f'(x) = f(x^\mu - c^\mu) = f(x) - c^\mu\partial_\mu f(x) & \Rightarrow P_\mu = -i\partial_\mu \\
\text{Lorentz}: & f'(x) = f(x^\mu) - w^\mu_\nu x^\nu\partial_\mu f(x) & \Rightarrow M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
\text{Dilatations}: & f'(x) = f(x^\mu) - \lambda x^\mu\partial_\mu f(x) & \Rightarrow D = -ix^\mu\partial_\mu \\
\text{Conf. boosts}: & f'(x) = f(x) - (2b^\nu x_\nu x^\mu\partial_\mu - x^2 b^\mu\partial_\mu)f(x) & \Rightarrow K_\mu = -i(2x_\mu x^\rho\partial_\rho - x^2\partial_\mu)
\end{array} \tag{2.23}$$

At this point we can straightforwardly compute the commutation rules using the explicit differential representation:

$$\begin{aligned}
&[D, P_\mu] = iP_\mu \\
&[D, K_\mu] = -iK_\mu \\
&[K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu} D - M_{\mu\nu}\right) \\
&[M_{\mu\nu}, P_\rho] = -i\left(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu\right) \\
&[M_{\mu\nu}, K_\rho] = -i\left(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu\right) \\
&[M_{\mu\nu}, M_{\rho\sigma}] = -i\left(M_{\mu\rho}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma} + M_{\nu\sigma}\eta_{\mu\rho}\right) \\
&[D, M_{\mu\nu}] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = [D, D] = 0
\end{aligned} \tag{2.24}$$

It is more convenient to redefine the generators $G \to iG$ to get ride of i's factors. The commutation relation become

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\rho\mu} - \eta_{\mu\sigma} M_{\rho\nu}\,, \\
&[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu\,, \\
&[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu\,, \\
&[D, P_\mu] = P_\mu\,, \\
&[D, K_\mu] = -K_\mu\,, \\
&[K_\mu, P_\nu] = 2\eta_{\mu\nu} D - 2M_{\mu\nu}\,.
\end{aligned} \tag{2.25}$$

In Euclidean signature, the conformal algebra is isomorphic to the algebra of $SO(d+1,1)$.[6] This is shown by the mapping

$$\mathcal{J}_{d+1\,\mu} = \left(P_\mu - K_\mu\right)/2\,, \quad \mathcal{J}_{d+2\,\mu} = \left(P_\mu + K_\mu\right)/2\,, \mathcal{J}_{\mu\nu} = M_{\mu\nu}\,, \quad \mathcal{J}_{d+1\,d+2} = D\,, \tag{2.26}$$

which satisfies the $SO(d+1,1)$ commutation relations

$$[\mathcal{J}_{AB}, \mathcal{J}_{CD}] = \eta_{BC}\mathcal{J}_{AD} - \eta_{AC}\mathcal{J}_{BD} + \eta_{BD}\mathcal{J}_{CA} - \eta_{AD}\mathcal{J}_{CB}\,, \tag{2.27}$$

where η_{AB} is the Lorentzian metric on $\mathbb{R}^{d+1,1}$.

Exercise 2.6. *Verify the commutations relations Eq. (2.27).*

2.4 Algebra of the Conformal Symmetry - II

There is an alternative and less painful way to extract the algebra of the conformal group. Earlier we introduced the generators of a given symmetry as the integral of the conserved currents. In these lectures we always assume the existence of a conserved, traceless stress

[6] In Lorentzian signature it is $SO(d,2)$.

tensor. If we have a conformal transformation with parameter ϵ^μ then the associated charge is

$$Q_\epsilon = \int_\Sigma dS_\mu \epsilon_\nu T^{\mu\nu} \tag{2.28}$$

where Σ is the boundary of some region $\Sigma = \partial B$ and dS_μ is the normal to the surface. We can then consider the action of this conserved charge on a given operator. We have nor defined how operators transform under the conformal transformation, however there is a particular case in which we don't need this information, i.e. if we consider the action on $T^{\mu\nu}$ itself. Because $\delta_\epsilon T^\mu \nu$ must have the same scaling dimension of $T^\mu \nu$ and should not spoil the symmetry, tracelessness or conservation, we must have

$$[Q_\epsilon, T^{\mu\nu}] = a_0 \epsilon^\rho \partial_\rho T^{\mu\nu} + a_1 (\partial_\rho \epsilon^\rho) T^{\mu\nu} + a_2 \partial_\rho \epsilon^\mu T^{\rho\nu} + a_3 \partial^\mu \epsilon_\rho T^{\rho\nu} \tag{2.29}$$

with $a_2 = -a3$ imposed by tracelessness and $a_0 = a_1 = -a_2$ imposed by conservation. We can then assume the usual transfomation property under translation

$$[Q_{c^\mu \equiv \text{cost}}, T^{\mu\nu}] = c^\rho \partial_\rho T^{\mu\nu} \tag{2.30}$$

to fix the overall normalization $a_0 = 1$. At this point we can constract with ϵ'^μ and integrate over a surface Σ' to obtain:

$$[Q_\epsilon, Q_{\epsilon'}] = Q_{-[\epsilon, \epsilon']} \tag{2.31}$$

where $[\epsilon, \epsilon'] = (\partial \cdot \epsilon)\epsilon' - (\partial \cdot \epsilon')\epsilon$ is the Lie bracket.

3 Quantum CFTs

3.1 Representation on Fields

Let us now consider the representation of the conformal algebra on a set of fields collectively called Φ_α. In order to construct a general representation we need to compute the action of the generator on $\Phi_\alpha(x)$. As a first step we compute the action of the stability group $x = 0$ on $\Phi_\alpha(0)$. Once this is known we can generalize the construction in the following way (**[?**]):

$$\Phi_\alpha(x) = e^{-iPx} \Phi_\alpha(0) e^{iPx} \quad \Rightarrow \quad [G, \Phi_\alpha(x)] = e^{-iPx} [\widetilde{G}, \Phi_\alpha(0)] e^{iPx} \tag{3.1}$$

where, making use of the Baker-Campbell-Hausdorff expansion, we have

$$\widetilde{G} \doteqdot e^{iPx} G e^{-iPx} = \sum_n \frac{(i)^n}{n!} x_{\mu_1} \dots x_{\mu_n} [P^{\mu_1}, [P^{\mu_2}, \dots [P^{\mu_n}, G] \dots]] \tag{3.2}$$

The resummation of the above series is straightforward whenever $[P_\mu, G] \propto P_\nu$. In this case the infinite series can be truncated to the linear order. This happens for the generators

D, $M_{\mu\nu}$. For what concerns K_μ, the series must be extended to the second term. Finally,

$$\begin{aligned}
\widetilde{D} &= D + x_\mu P^\mu \,, \\
\widetilde{M}_{\mu\nu} &= M_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \,, \\
\widetilde{K}_\mu &= K_\mu + 2x_\mu D + 2x^\rho M_{\rho\mu} + 2x_\mu x^\rho P_\rho - x^2 P_\mu \,,
\end{aligned} \tag{3.3}$$

and we obtain the generator representation on fields:

$$\begin{aligned}
&[P_\mu, \Phi_\alpha(x)] = i\partial_\mu \Phi_\alpha(x) \,, \\
&[D, \Phi_\alpha(x)] = i\Delta\Phi_\alpha(x) + ix^\mu\partial_\mu\Phi_\alpha(x) \,, \\
&[M_{\mu\nu}, \Phi_\alpha(x)] = i(S_{\mu\nu})_{\alpha\beta}\Phi_\beta(x) - i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi_\alpha(x) \,, \\
&[K_\mu, \Phi_\alpha(x)] = i\mathcal{K}_\mu\Phi_\alpha(x) - 2ix_\mu\Delta\Phi_\alpha(x) + 2x^\rho i(S_{\rho\mu})_{\alpha\beta}\Phi_\beta(x) + i\left(2x_\mu x^\rho\partial_\rho\Phi_\alpha(x) - x^2\partial_\mu\Phi_\alpha(x)\right) \,,
\end{aligned} \tag{3.4}$$

where we have introduced the quantities

$$[D, \Phi_\alpha(0)] = i\Delta\Phi_\alpha(0) \,, \qquad [M_{\mu\nu}, \Phi_\alpha(0)] = i(S_{\mu\nu})_{\alpha\beta}\Phi_\beta(0) \,, \qquad [K_\mu, \Phi_\alpha(0)] = i\mathcal{K}_\mu\Phi_\alpha(0) \,. \tag{3.5}$$

Let us discuss the representations of the stability group of $x = 0$. In a given irreducible representation of the Lorentz group the generator K_μ vanishes identically. Indeed, D being a Lorentz scalar, by Shur's Lemma, it must to be proportional to the identity, since it commutes with all the generators of the representation. Thus, the commutation relation $[D, K_\mu] = -iK_\mu$ requires $K_\mu = 0$.
Let now start from a reducible *finite dimensional* Lorentz representation. In this case K_μ can be different form zero but it must be is nilpotent. This because K_μ acts as lowering operator for the dilatation: if an operator O has dimension Δ than $[K_\mu, O]$ has dimension $\Delta - 1$. Since the representation is finite dimensional by assumptions, the repeated action of K_μ must give zero after a finite number of steps.
Let us now consider representation of the entire conformal group. The generator P_μ acts as raising operator with respect to the dilatations generator. Hence we conclude that representations on fields cannot be finite dimensional because P_μ cannot be nilpotent. The same conclusion can be achieved requiring the representation to be unitary since the conformal group is non-compact and cannot have unitary finite dimensional representation, as already discussed in Section **??**.
Following to the above reasoning we can classify representations of the conformal group according to the Lorentz quantum number and the scaling dimension of the lowest dimension operator appearing in the representation. Such an operator is called *primary field* while all the other operators in the representation can be obtained acting with P_μ and are denoted *descendants*.

Integrating the above infinitesimal transformations one can see that primary operators transforms as

$$\mathcal{O}'^{\mathbf{i}}_{\Delta,r}(x') = \mathcal{F}^{\mathbf{i}}{}_{\mathbf{j}}\mathcal{O}^{\mathbf{j}}_{\Delta,r}(x) \,, \quad \mathcal{F} = \frac{1}{\Omega(x)^\Delta}\mathcal{R}[\Lambda^\mu{}_\nu(x)] \,, \tag{3.6}$$

where $\mathcal{R}[\Lambda^\mu{}_\nu(x)]$ is the matrix representing the finite rotation $\Lambda^\mu{}_\nu(x)$ in the representation r. [7]

[end of lecture 4]

3.2 EX: two and three point functions

In order to keep the notation compact, we denote a generic coordinate transformation by $x'^\mu = f^\mu(x)$ and its inverse by $x''^\mu = (f^{-1})^\mu(x)$.
Let us begin considering the correlation functions of two scalar primary fields O_1 and O_2 with scaling dimensions Δ_1 and Δ_2. Inserting in the correlation function the unitary operator implementing the coordinate rescaling $x'^\mu = \lambda^{-1}x^\mu$ we obtain the relation[8]

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \langle O_1'(x_1)\, O_2'(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle O_1(\lambda x_1)\, O_2(\lambda x_2)\rangle\,. \tag{3.8}$$

Poincaré invariance of the scalar fields forces the result to depend only on the combination $x_{12} = |x_1 - x_2|$ and the above relation states that it must be an homogenous function of degree $-\Delta_1 - \Delta_2$. The only function with this properties is $x_{12}^{-\Delta_1-\Delta_2}$. Considering in addition conformal boosts and recalling the transformation properties of x_{12} (see for instance eq. (**??**))

$$\begin{aligned}
\langle O_1'(x_1)\, O_2'(x_2)\rangle &= \frac{1}{(1-2b\cdot x_1 + b^2x_1^2)^{\Delta_1}(1-2b\cdot x_2+b^2x_2^2)^{\Delta_2}}\langle O_1(x_1'')\, O_2(x_2'')\rangle \\
&= \frac{1}{(1-2b\cdot x_1 + b^2x_1^2)^{\Delta_1}(1-2b\cdot x_2+b^2x_2^2)^{\Delta_2}}\frac{C_{12}}{(x_{12}'')^{\Delta_1+\Delta_2}} \\
&= \frac{C_{12}}{(x_{12})^{\Delta_1+\Delta_2}}\frac{(1-2b\cdot x_1+b^2x_1^2)^{\frac{\Delta_1+\Delta_2}{2}}(1+2b\cdot x_2+b^2x_2^2)^{\frac{\Delta_1+\Delta_2}{2}}}{(1-2b\cdot x_1 + b^2x_1^2)^{\Delta_1}(1-2b\cdot x_2+b^2x_2^2)^{\Delta_2}}
\end{aligned} \tag{3.9}$$

The above relation can be verified with a non vanishing constant C_{12} if and only if $\Delta_1 = \Delta_2$. In all the other cases $C_{12} = 0$. Hence:

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta}}\,, & \text{if } \Delta_1 = \Delta_2 = \Delta \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

In general there can be non vanishing correlation function between non identical operators: it is sufficient they have the same scaling dimension. On the other hand in unitary theories we can diagonalize the subspaces of operators with equal dimension and rescale the fields such that

$$\langle O_i(x_1)\, O_j(x_2)\rangle = \begin{cases} \frac{1}{|x_1-x_2|^{2\Delta}}\,, & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{3.11}$$

[7] Although we write the l.h.s as $\mathcal{O}'$ (as is customary), it's important to remember that $\mathcal{O}$ and $\mathcal{O}'$ represent the same operator.

[8] Here we used the fact that the vacuum is invariant and therefore:

$$f(x_i) = \langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \langle U_g\mathcal{O}_1(x_1)U_g^{-1}\ldots\mathcal{U}_gO_n(x_n)U_g^{-1}\rangle = \langle \mathcal{O}_1'(x_1)\ldots\mathcal{O}_n'(x_n)\rangle = \tag{3.7}$$

Conformal symmetry also fixes the structure of three point function. Analogously as before coordinate rescaling restricts the generic form of the correlation function to

$$\langle O_1(x_1)\, O_2(x_2) O_3(x_3)\rangle = \frac{C_{123}}{x_{12}^a x_{13}^b x_{23}^c}\,, \qquad \text{and } a+b+c = \Delta_1+\Delta_2+\Delta_3 \tag{3.12}$$

In principle there could be a sum on different terms but we will see in the following that the coefficients a, b, c are completely fixed. Indeed conformal boost covariance forces the following relation:

$$\begin{aligned}&\langle O_1'(x_1)\, O_2'(x_2) O_3'(x_3)\rangle = \\ &= \frac{C_{123}}{x_{12}^a x_{13}^b x_{23}^c}\frac{(1+2b\cdot x_1+b^2x_1^2)^{\frac{a+b}{2}}(1+2b\cdot x_2+b^2x_2^2)^{\frac{a+c}{2}}(1+2b\cdot x_3+b^2x_3^2)^{\frac{b+c}{2}}}{(1+2b\cdot x_1+b^2x_1^2)^{\Delta_1}(1+2b\cdot x_2+b^2x_2^2)^{\Delta_2}(1+2b\cdot x_3+b^2x_3^2)^{\Delta_3}}\,.\end{aligned} \tag{3.13}$$

The only solution satisfying the above constraint for any choice of b_μ is:

$$\langle O_1(x_1)\, O_2(x_2) O_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1-\Delta_2+\Delta_3} x_{23}^{-\Delta_1+\Delta_2+\Delta_3}}\,. \tag{3.14}$$

3.3 Generaliazation

The general form of correlation functions that we will need in this course can be proven to be:

$$\begin{aligned}
\langle O_i(x_1)\, O_i(x_2)\rangle &= \frac{1}{|x_1-x_2|^{2\Delta_i}}\\
\langle t^{\mu_1\dots\mu_l}(x_1)\, t^{\nu_1\dots\nu_l}(x_2)\rangle &= \frac{1}{|x_1-x_2|^{2\Delta_t}}\left(\left(\eta^{\mu_1\nu_1}-2\frac{x_{12}^{\mu_1}x_{12}^{\nu_1}}{x_{12}^2}\right)\dots\left(\eta^{\mu_l\nu_l}-2\frac{x_{12}^{\mu_l}x_{12}^{\nu_l}}{x_{12}^2}\right)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\text{symmetrized - traces}\right)\\
\langle O_1(x_1) O_2(x_2) t^{\mu_1\dots\mu_\ell}(x_3)\rangle &= C_{OOt}\frac{Z_{123}^{\mu_1}\dots.Z_{123}^{\mu_\ell}-\text{traces}}{x_{12}^{(\Delta_1+\Delta_2-\Delta_3+\ell)/2}x_{13}^{(\Delta_1-\Delta_2+\Delta_t-\ell)/2}x_{23}^{(\Delta_2+\Delta_t-\Delta_1-\ell)/2}}\\
Z_{ijk}^\mu &= \frac{x_{ik}^\mu}{x_{ik}^2}-\frac{x_{jk}^\mu}{x_{jk}^2}
\end{aligned} \tag{3.15}$$

where O_i are scalar fields, while $t^{\mu_1\dots\mu_\ell}$ is a traceless symmetric tensor with ℓ indexes.

3.3.1 4pt functions

Finally let us consider 4pt functions, which as mentioned in Sec. ?? play fundamental role in the conformal bootstrap. Focusing here on the case of scalars, the 4pt function must take the general form

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle = \mathbf{K}_4\, g(u,v)\,. \tag{3.16}$$

The prefactor $\mathbf{K}_4 = \mathbf{K}_4(\Delta_i, x_i)$ is given by

$$\mathbf{K}_4 = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} , \tag{3.17}$$

where $\Delta_{ij} = \Delta_i - \Delta_j$. This prefactor by itself transforms correctly under conformal transformations. The remaining part of the correlator, $g(u, v)$, must be a function of two *cross ratios* u, v:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} , \tag{3.18}$$

which are invariant under all conformal transformations.

While no further information about $g(u, v)$ can be obtained from conformal invariance alone, it can in fact be computed in terms of the CFT data using additional tools such as the OPE and the conformal blocks. Let us see how in the next sections.

[end of lecture 5]

3.4 Radial-Quantization

In a conformal invariant quantum field theory the Hilbert space of states can be organized in irreducible representations of the conformal group. We would like now to understand the structure of local operators that generate the Hilbert space, very much like in a free theory the Fock space can be obtained by repetitive action of the ladder operators.

The first step in this direction is to define what we mean by evolution operator in a CFT. In ordinary QFT we are used to consider time evolution as generated by the Hamiltonian $H = P^0$, namely the time component of the translation generator P^μ. By doing this we can evolve some initial condition (or state) defined on a spacial slice at some time t_1. to an evolved state at time t_2. This is very convenient of we choose eigenstates of P^μ as basis of the Hilbert space.

In CFTs however the situation is somewhat different. We have seen that a irreps are built from an highest state, the primary state, which is not an eigenstate of translations. This means that the usual evolution operator would evolve a primary state into a mixture of primary and descendants, which s inconvenient. In order to overcome this issue, it is amenable to choose the operator D as the evolution operator. By doing this choice, we are effectively choosing a different foliation of the spacetime: instead of fixed time slices, we are foliating by spheres with fixed radious.

Once we have made this switch of prospective, it is immediate to identify the operator content a CFT. To proceed we can make use of the Path Integral formalism.

3.5 Digression on Path Integral

In order to understand what a Quantum Field Theory is, let us first first understand the concept of fields and what they represent. Classical Fields can be though as function of some space-time coordinates x_μ defined on a manifold $\mathcal{M}$. The latter can be some space $\mathcal{S} \subset \mathbb{R}^n$, a product of a manifold and an interval $\mathcal{N} \times \mathcal{I}$, or a generic Riemannain manifold.

A field is then a map from $\mathcal{M}$ to some other space $\mathcal{X}$, called target space:

$$\begin{aligned} \phi : \quad \mathcal{M} &\longrightarrow \mathcal{X} \\ x_\mu &\longrightarrow \phi(x) \,. \end{aligned} \tag{3.19}$$

The target space $\mathcal{X}$ can be of different kinds: $\mathbb{R}, \mathbb{C}$ for real or complex fields, a coset space $G\ H$ for σ-models, a tangent boudle for gauge fields, etc.

The space of all possible fields configurations, namely the space of all the functions of the form Eq. (3.19), will be denoted $\mathcal{C}$. It a very complicated and infinite dimensional space, whose characterisation goes beyond the scope of these notes. Nevertheless we will assume that it is possible to define a functional on $\mathcal{C}$ that associates to any field configuration a real number:

$$\begin{aligned} S : \quad \mathcal{C} &\longrightarrow \mathbb{R} \\ \phi &\longrightarrow S[\phi] \,. \end{aligned} \tag{3.20}$$

Such a functional is the *action*.

Let us now introduce the Path integral formulation of a quantum field theory. This is generalization of the sum over paths. The first trivial generalization is to replace the action as a function of coordinates to a functional over fields. In quantum mechanics the coordinate $\hat{x}$ is an operator that gives the position, and is a function of time, which instead is not an operator. In QFT space coordinates and space are treated on equal footing: they are the labels of some point p on our manifold $\mathcal{M}$. The quantum objects are the fields $\phi(x,t)$, therefore the Path integral should integrate over the field configurations:

$$\mathscr{D}_{QM}[x(t)] \longrightarrow \mathscr{D}_{QFT}[\phi(x,t)] \tag{3.21}$$

The integral is taken over the space of field configurations $\mathcal{C}$. Finally the boundary conditions. In order to understand them we must take a step back and understand what does the Path Integral represent in QFT. We have seen that in QM the Path Integral defines the probability amplitude to transit from a state at x_i at time t_i to a state at time x_f at time t_f.

In order to have the same picture in QFT, we need first to define the notion of evolution. Given a manifold $\mathcal{M}$, we can choose a foliation $\mathcal{M} \simeq \Sigma \times \mathbb{R}$ or $\mathcal{M} \simeq \Sigma \times \mathcal{I}$, where $\mathcal{I}$ is an inteval, for instance $[0, T]$. Choosing a foliation correspond to choose a direction which we call time, and some space slicing that we call Σ. In Conformal Field Theories a convenient foliation is $S^{d-1} \times \mathbb{R}_+$.

Once we have defined what we mean by time, we can need to define what data we want to evolve. On each slice Σ we can define a Hilbert space $\mathcal{H}$: the Hilbert space is the set of states living on that slice. For instance if our theory is defined by a fundamental field $\phi(x,t)$, than the states are defined by the configuration of the field on that slice,

$$\phi(x \in \Sigma, t = t_i) \tag{3.22}$$

The wave function of a generic state $|\Psi\rangle$ on that slice is then:

$$\langle \phi | \Psi \rangle = \Psi[\phi(x, t_i)] \tag{3.23}$$

where on the rhs we have some function of the field ϕ. In conclusion, the QFT Path integral

$$\langle \phi_f | U(t_f - t_i) | \phi_i \rangle = \int_{\substack{\phi(x,t_i)=\phi_i \\ \phi(x,t_f)=\phi_f}} \mathscr{D}[\phi(x,t)] e^{iS[\phi]} \tag{3.24}$$

describes the probability to transit from a field configuration ϕ_i at time t_i to a field configuration ϕ_f at time t_f.Here $U(t)$ is the evolution operator. By stripping off the final boundary condition we essencially define the evolved state:

$$U(t_f - t_i) | \phi_i \rangle = \int_{\substack{\phi(x,t_i)=\phi_i \\ \phi(x,t_f)=\text{free}}} \mathscr{D}[\phi(x,t)] e^{iS[\phi]} \tag{3.25}$$

since by definition the above state has the correct overlapping with any other state of the theory. Finally, we can use the Path Integral to define the vacuum state of the theory: if we take the limit of infinite negative time, the dominant contribution to the above expression is from the lowest energy state of the theory[9]:

$$|0\rangle_{\text{un}} = \lim_{t_i \to -\infty} \int_{\substack{\phi(x,t_i)=\phi_i \\ \phi(x,t_f)=\text{free}}} \mathscr{D}[\phi(x,t)] e^{iS[\phi]} \tag{3.26}$$

The above definition gives the unnormalized vacum state. By dividing by its norm one gets the nornmalized one. An important result is that the normalized vacuum state doesn't depend on the choice of the initial condition ϕ_i. We can then leave it arbitray.

3.6 States $\Leftrightarrow$ Operators correspondence

We are now ready to relate states and operators in a CFT. Let us start from the vacuum. Let us first start by understanding what is a state in radial quantization. If our CFT is

[9] This is best seen by passing in the Euclidean formulation of QFT, where the oscillating phases become exponential dumping. Then, by inserting a complete set of energy states one obtain the result.

defined through a set of operators, then a state is determined by the field configurations on a given slice of our spacetime foliation, hence a sphere. Given such state, we can use the dilatation operator to evolve it back and forth, namely contract or expand the sphere. The parameter of the dilatation will be the ratio of the two radiai r/r_i. According to the discussion of the previous section, by sending $r_i \to 0$, we are providing a definition of the vacuum in radial quantization (RQ) on sphere of radius r

$$|0\rangle_r = \int_{B_r} \mathscr{D}[\mathcal{O}_{\text{CFT}}] e^{iS_{\text{CFT}}} \tag{3.27}$$

where the integral is performed inside a ball B_r of radius r. One can check that this state is invariant under all the symmetries. Now let us make things more interesting by inserting an operator $\mathcal{O}(0)$ at the origin. We'll denote $|\mathcal{O}\rangle$ the state created in this way:

$$|\mathcal{O}\rangle_r = \int_{B_r} \mathscr{D}[\mathcal{O}_{\text{CFT}}] \mathcal{O}(0) e^{iS_{\text{CFT}}} \tag{3.28}$$

This state has the same quantum number of the operator and in particular is a primary state. We can interpret it as the action of the operator on the vacuum $\mathcal{O}(0)|0\rangle$. We can verify that this is the case by computing the action of a generator:

$$Q_\epsilon \mathcal{O}(0)|0\rangle = [Q_\epsilon, \mathcal{O}(0)]\,|0\rangle = \tag{3.29}$$

Similarly we can ask what is the effect of inserting in the Path Integral an operator at some non zero point $\mathcal{O}(x)$. This will create a state on sphere encircling the point x:

$$|\mathcal{O}(x)\rangle_r = e^{-iP\cdot x}\mathcal{O}(0)e^{iP\cdot x}|0\rangle_r = |\mathcal{O}(0)\rangle_r + ix\cdot P|\mathcal{O}(0)\rangle_r + \dots \tag{3.30}$$

which is a infinite superposition of primary and descendands.

The bottom line of the above argument is that operators inserted at some generic point inside a ball define a state on the boundary of the ball. No insertion (equivalently inserting the identity) correspond to the vacuum.

Let us now discuss the reverse logic. Given states defined on spheres, how do we reconstruct the operators? An operator is completely defined by its matrix elements between other states of the theory, hence by its correlation functions. Consider a correlation function containing n operators:

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle \tag{3.31}$$

Say we want to define the operator $\mathcal{O}_i(x_i)$. By making a translation we can always place set $x_i = 0$. Then we can cut a hole around x_i. We can then replace the presence of the operator with the state defined on the sphere:

$$\langle \mathcal{O}_i|\text{other operators}\rangle \tag{3.32}$$

This operation can be iterated for any operator, thus fully defining the CFT operators in terms of states.

3.7 The Operator Product Expansion (OPE)

If we insert two operators $\mathcal{O}_i(x)\mathcal{O}_j(0)$ inside a ball and perform the path integral over the interior, we get some state on the boundary. Because every state is a linear combination of primaries and descendants, we can decompose this state as

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle = \sum_k C_{ijk}(x,P)\mathcal{O}_k(0)|0\rangle, \tag{3.33}$$

where k runs over primary operators and $C_{ijk}(x,P)$ is an operator that packages together primaries and descendants in the k-th conformal multiplet.

Eq. (3.33) is an exact equation that can be used in the path integral, as long as all other operators are outside the sphere with radius $|x|$. Using the state-operator correspondence, we can write

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12},\partial_2)\mathcal{O}_k(x_2), \qquad \text{(OPE)} \tag{3.34}$$

where it is understood that (3.34) is valid inside any correlation function where the other operators $\mathcal{O}_n(x_n)$ have $|x_{2n}| \geq |x_{12}|$. Eq. (3.34) is called the Operator Product Expansion (OPE).

We could alternatively perform radial quantization around a different point x_3, giving

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C'_{ijk}(x_{13},x_{23},\partial_3)\mathcal{O}_k(x_3), \tag{3.35}$$

where $C'_{ijk}(x_{13},x_{23},\partial_3)$ is some other differential operator. The form (3.34) is usually more convenient for computations, but the existence of (3.35) is important. It shows that we can do the OPE between two operators whenever it's possible to draw any sphere that separates the two operators from all the others.

We are being a bit schematic in writing the above equations. It's possible for all the operators to have spin. In this case, the OPE looks like

$$\mathcal{O}_i^a(x_1)\mathcal{O}_j^b(x_2) = \sum_k C_{ijkc}^{ab}(x_{12},\partial_2)\mathcal{O}_k^c(x_2), \tag{3.36}$$

where a,b,c are indices for (possibly different) representations of SO(d).

[end of lecture 6]

3.8 Consistency with Conformal Invariance

Conformal invariance strongly restricts the form of the OPE. For simplicity, suppose $\mathcal{O}_i$, $\mathcal{O}_j$, and $\mathcal{O}_k$ are scalars.

Exercise 3.1. *By acting on both sides of (3.33) with D, prove that $C_{ijk}(x,\partial)$ has an expansion of the form*

$$C_{ijk}(x,\partial) \quad \propto \quad |x|^{\Delta_k-\Delta_i-\Delta_j}\left(1+\#x^\mu\partial_\mu+\#x^\mu x^\nu\partial_\mu\partial_\nu+\#x^2\partial^2+\dots\right). \tag{3.37}$$

This is just a fancy way of saying we can do dimensional analysis and that $\mathcal{O}_i$ has length-dimension $-\Delta_i$. We're also implicitly using rotational invariance by contracting all the indices appropriately. We could have proved this too by acting with $M_{\mu\nu}$.

We get a more interesting constraint by acting with K_μ. In fact, consistency with K_μ completely fixes C_{ijk} up to an overall coefficient. In this way, we can determine the coefficients in (3.37).

This computation is a little annoying (exercise!), so here's a simpler way to see why the form of the OPE is fixed, and to get the coefficients in (3.37). Take the correlation function of both sides of (3.34) with a third operator $\mathcal{O}_k(x_3)$ (we will assume $|x_{23}|\geq|x_{12}|$, so that the OPE is valid),

$$\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle \quad = \quad \sum_{k'}C_{ijk'}(x_{12},\partial_2)\langle\mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle. \tag{3.38}$$

The three-point function on the left-hand side is fixed by conformal invariance, and is given in (**??**). We can choose an orthonormal basis of primary operators, so that $\langle\mathcal{O}_k(x_2)\mathcal{O}_{k'}(x_3)\rangle = \delta_{kk'}x_{23}^{-2\Delta_k}$. The sum then collapses to a single term, giving

$$\frac{f_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k}x_{23}^{\Delta_j+\Delta_k-\Delta_i}x_{31}^{\Delta_k+\Delta_i-\Delta_j}} \quad = \quad C_{ijk}(x_{12},\partial_2)x_{23}^{-2\Delta_k}. \tag{3.39}$$

This determines C_{ijk} to be proportional to f_{ijk}, times a differential operator that depends only on the Δ_i's. The operator can be obtained by matching the small $|x_{12}|/|x_{23}|$ expansion of both sides of (3.39).

Exercise 3.2. *Consider the special case $\Delta_i=\Delta_j=\Delta_\phi$, and $\Delta_k=\Delta$. Show*

$$\begin{aligned}
C_{ijk}(x,\partial) &= f_{ijk}x^{\Delta-2\Delta_\phi}\left(1+\frac{1}{2}x\cdot\partial+\alpha x^\mu x^\nu\partial_\mu\partial_\nu+\beta x^2\partial^2+\dots\right),\\
\alpha &= \frac{\Delta+2}{8(\Delta+1)}, \quad \textit{and} \quad \beta=-\frac{\Delta}{16(\Delta-\frac{d-2}{2})(\Delta+1)}.
\end{aligned} \tag{3.40}$$

For arbitrary operator dimensions one has instead

$$\begin{aligned}
C_{ijk}(x,\partial) = \frac{f_{ijk}}{x^{\Delta_1+\Delta_2-\Delta_3}}\Bigg(1&+\frac{\Delta_3-\Delta_1+\Delta_2}{2\Delta_3}x^\mu\partial_\mu\\
&+\frac{1}{8}\frac{(\Delta_3-\Delta_1+\Delta_2)(\Delta_3-\Delta_1+\Delta_2+2)}{\Delta_3(\Delta_3+1)}x^\mu x^\nu\partial_\mu\partial_\nu\\
&-\frac{1}{16}\frac{(\Delta_3-\Delta_1+\Delta_2)(\Delta_3+\Delta_1-\Delta_2)}{\Delta_3(\Delta_3+1-d/2)}x^\mu x^2\Box\partial_\mu+\dots.\Bigg),
\end{aligned} \tag{3.41}$$

Exercise 3.3. *Show that the 3pt function of identical scalars and a third operator can only be non-vanishing if the operators is a traceless symmetric tensor.*

3.9 Interlude: Free theories

Consider a theory of a free real scalars ϕ. This theory is scale clearly scale invariant at classical level and quantum level since there are no interactions. This can be shown by computing the stress tensor:

$$T_{\mu}\nu =: \partial_\mu\phi\partial_\nu\phi : -\frac{1}{2}\eta_{\mu\nu} : (\partial_\mu\phi)^2 : \tag{3.42}$$

Here the notation :: means normal ordered. If we take the trace of the above expression we see that it doesn't vanish, however it is proportional to

$$T^\mu_\mu = (1 - d/2) : (\partial_\mu\phi)^2 : \tag{3.43}$$

which is a total derivatives in the equations of motion $\Box\phi = 0$, $: (\partial_\mu\phi)^2 :\sim \partial_\mu : \phi\partial^\mu\phi := \partial_\mu V^\mu$. Here V^μ is the Virial current. This is one of the cases in which the Energy momentum tensor can indeed be modified to obtain a symmetric, conserved and traceless operator. This because the Virial current is in fact a divergence: $V^\mu = \frac{1}{2}\partial^\mu : \phi^2 := \partial_\nu\sigma^{\mu\nu}$, with $\sigma^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu} : \phi^2 :$. Then one can show that an improved tensor T', defined as

$$\begin{aligned}
\sigma^{\mu\nu}_+ &= \frac{1}{2}\left(\sigma^{\mu\nu} + \sigma^{\nu\mu}\right)\\
X^{\lambda\rho\mu\nu} &= \frac{2}{d-2}\left(\eta^{\lambda\rho}\sigma^{\mu\nu}_+ - \eta^{\lambda\mu}\sigma^{\rho\nu}_+ - \eta^{\rho\mu}\sigma^{\lambda\nu}_+ + \eta^{\mu\nu}\sigma^{\lambda\rho}_+ + \frac{1}{d-1}\left(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\rho\nu}\eta^{\lambda\mu}\sigma^{\gamma}_{+\gamma}\right)\right)\\
T'_{\mu\nu} &= T_{\mu\nu} + \partial^\lambda\partial^\rho X_{\lambda\rho\mu\nu}
\end{aligned} \tag{3.44}$$

satisfies the required properties. In our case this becomes:

$$T'_{\mu\nu} =: \partial_\mu\phi\partial_\nu\phi : -\frac{1}{4(d-1)}\left((d-2)\partial^\mu\partial^\nu + \eta^{\mu\nu}\Box\right) : \phi^2 : \tag{3.45}$$

Now that we have established conformal invariance, we can compute a few correlation functions and check they indeed satisfy the general construction we made. The starting point is

$$\phi(x) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}2k_0}\left(a(\vec{k})e^{ikx} + a^\dagger(\vec{k})e^{-ikx}\right) \tag{3.46}$$

Then the 2pt function is

$$\begin{aligned}
\langle\phi(x)\phi(y)\rangle \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle &= \int \frac{d^{d-1}k}{(2\pi)^{d-1}2k_0}\frac{d^{d-1}p}{(2\pi)^{d-1}2p_0}\langle 0|a(\vec{p})a^\dagger(\vec{k})|0\rangle e^{ipx-iky}\\
&= \int \frac{d^{d-1}k}{(2\pi)^{d-1}2k_0}e^{ik(x-y)} = \frac{1}{|x-y|^{d-2}}
\end{aligned} \tag{3.47}$$

This means that the scalar ϕ behaves as an opertor of dimensions $\Delta_\phi = (d-2)/2$. At this point we can compute any other correlation function using the Wick theorem:

$$T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} =: \phi(x_1)\phi(x_2)\dots\phi(x_n) : + \overbrace{\phi(x_1)\phi}(x_2) : \dots\phi(x_n) : + \dots \tag{3.48}$$

A simple case is the case of four scalars:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\left(1+u^{\Delta_\phi/2}+\left(\frac{u}{v}\right)^{\Delta_\phi/2}\right) \tag{3.49}$$

Similarly one can compute for instance the 3pt function

$$\langle\phi(x_1)\phi(x_2) : \phi^2(x_3) :\rangle = \frac{1}{x_{12}^{2\Delta_\phi}x_{13}^{2\Delta_\phi}} \tag{3.50}$$

Exercise 3.4. *Compute the three point function $\langle\phi(x_1)T^{\mu\nu}(x_2)\phi(x_3)\rangle$ and show that it verifies the Ward identities.*

Exercise 3.5. *Compute the four point point function $\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle$, where $O =: \phi^2 :$.*

3.10 Conformal blocks

Consider a 4pt function of four primary *scalar* operators $\phi_i(x_i)$ with $i = 1, \ldots, 4$. This 4pt function can be computed by applying the OPE of Eq. (3.34) to two pairs of fields. For definiteness we fix here the pairing $\phi_1(x_1)\phi_2(x_2)$ and $\phi_3(x_3)\phi_4(x_4)$. This is referred to as "the (12)-(34) OPE channel", to distinguish it from other pairings which will play a role when we discuss crossing. This gives an expansion

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, \mathrm{W}_{\mathcal{O}}\,, \tag{3.51}$$

where $\mathrm{W}_{\mathcal{O}} \equiv \mathrm{W}_{\mathcal{O}}(x_i)$ are the *conformal partial waves* (CPWs) given by

$$\mathrm{W}_{\mathcal{O}} = \hat{f}_{12\mathcal{O}}(x_1, x_2, y, \partial_y)\hat{f}_{34\mathcal{O}}(x_3, x_4, y', \partial_{y'})\langle\mathcal{O}(y)\mathcal{O}(y')\rangle\,. \tag{3.52}$$

Since the 2pt function is diagonal, the summation is over the same operator $\mathcal{O}$ in both OPEs. It follows from conformal invariance of the OPE that each CPW transforms under the conformal transformations in the same way as the 4pt function itself. It is then conventional to split off the factor $\mathbf{K}_4$ defined in Eq. (3.17), so that we finally have

$$\mathrm{W}_{\mathcal{O}} = g^{\Delta_{12},\Delta_{34}}_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v)\,\mathbf{K}_4\,, \tag{3.53}$$

where $g^{\Delta_{12},\Delta_{34}}_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v)$ is called the conformal block. In part of the literature these two terms are used interchangeably. It represents the contribution of a primary $\mathcal{O}$ and all of its descendants to the 4pt function. As shown, it depends on the dimension and spin of the exchanged traceless symmetric primary $\mathcal{O}$, and also on the dimension differences Δ_{12}, Δ_{34} of the external scalars.[10] Comparing with Eq. (3.16), we thus have:

$$g(u,v) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g^{\Delta_{12},\Delta_{34}}_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v)\,. \tag{3.54}$$

[10]For definitiveness $\Delta_{ij} = \Delta_i - \Delta_j$. Sometimes we will omit the latter dependence, if it is clear from the context.

Eqs. (3.51) and (3.54) are referred to as the CPW decomposition and the conformal block decomposition.

Let us briefly discuss the regions of convergence of the considered expansions. If one works in the z conformal frame of Eq. (??) in Euclidean signature, then Eq. (3.52) defining the CPWs converges for $|z| < 1$, and the conformal block decomposition (3.54) is also seen to converge in this region, at least if the theory is unitary.

The above definition of conformal blocks via the conformal OPE is important in principle. In practice, there exist efficient approaches to compute the blocks which avoid needing explicit knowledge of the conformal OPE. They will be described below.

3.11 The Casimir equation

Let us consider the following alternative representation of CPWs. In radial quantization, as mentioned in Sec. ??, the above 4pt function is expressed as a scalar product of two states

$$\langle \phi_3(x_3)\phi_4(x_4)|\phi_1(x_1)\phi_2(x_2)\rangle \tag{3.55}$$

living on a sphere separating x_1, x_2 from x_3, x_4. The CPW then corresponds to inserting an orthogonal projector $\mathcal{P}_{\Delta,\ell}$ onto the conformal multiplet of $\mathcal{O}_{\Delta,\ell}$:

$$\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, \mathrm{W}_{\mathcal{O}} = \langle \phi_3(x_3)\phi_4(x_4)|\mathcal{P}_{\Delta,\ell}|\phi_1(x_1)\phi_2(x_2)\rangle\,. \tag{3.56}$$

For future reference, the projector can be written as

$$\mathcal{P}_{\Delta,\ell} = \sum_{\alpha,\beta=\mathcal{O},P\mathcal{O},PP\mathcal{O},\ldots} |\alpha\rangle G^{\alpha\beta}\langle\beta|\,, \tag{3.57}$$

where $G_{\alpha\beta} = \langle\alpha|\beta\rangle$ is the Gram matrix of the multiplet.

Furthermore, consider the quadratic Casimir[11]

$$\mathcal{C}_2 = \frac{1}{2}\mathcal{J}_{AB}\mathcal{J}^{BA}\,, \tag{3.58}$$

where $\mathcal{J}_{AB}$ are the $SO(d+1,1)$ generators, Eq. (2.27). Insert this operator into Eq. (3.56) right after $\mathcal{P}_{\Delta,\ell}$. The resulting expression can be computed in two ways. When we act with $\mathcal{C}_2$ on the left we have

$$\mathcal{P}_{\Delta,\ell}\,\mathcal{C}_2 = C_{\Delta,\ell}\mathcal{P}_{\Delta,\ell}\,, \tag{3.59}$$

where $C_{\Delta,\ell}$ is the quadratic Casimir eigenvalue:

$$C_{\Delta,\ell} = \Delta(\Delta - d) + \ell(\ell + d - 2)\,. \tag{3.60}$$

On the other hand, the action of $\mathcal{C}_2$ on the right can be computed using the representation of the conformal generators on primaries as first-order differential operators, mentioned in

[11]The quartic Casimir operator $\mathcal{C}_4 = \frac{1}{2}\mathcal{J}_{AB}\mathcal{J}^{BC}\mathcal{J}_{CD}\mathcal{J}^{DA}$ has also proved useful in some conformal block studies

Sec. **??**. We conclude that the CPW, and hence the conformal block, satisfies a second-order partial differential equation. The same conclusion can be reached using the OPE [**?**]. The actual form of this "Casimir equation" is most conveniently found using the embedding formalism [**?**]. In the $z, \overline{z}$ coordinates of Eq. (**??**) it takes the form

$$\mathcal{D}\, g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\overline{z}) = C_{\Delta,\ell}\; g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\overline{z})\,, \tag{3.61}$$

where

$$\begin{aligned}\mathcal{D} &= \mathcal{D}_z + \mathcal{D}_{\overline{z}} + 2(d-2)\frac{z\overline{z}}{z-\overline{z}}[(1-z)\partial_z - (1-\overline{z})\partial_{\overline{z}}]\,,\\ \mathcal{D}_z &= 2z^2(1-z)\partial_z^2 - (2+\Delta_{34}-\Delta_{12})z^2\partial_z + \tfrac{\Delta_{12}\Delta_{34}}{2}z\,.\end{aligned} \tag{3.62}$$

Moreover, the leading $z, \overline{z} \to 0$ behavior of the conformal block can be easily determined using the OPE, and this provides boundary conditions for Eq. (3.61). Considering the $x_{12}, x_{34} \to 0$ limit in Eq. (3.53) and using Eqs. (**??**) and (**??**), one obtains

$$g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\overline{z}) \underset{z,\overline{z}\to 0}{\sim} \mathcal{N}_{d,\ell}\,(z\overline{z})^{\frac{\Delta}{2}}\,\mathrm{Geg}_\ell\left(\frac{z+\overline{z}}{2\sqrt{z\overline{z}}}\right)\,, \tag{3.63}$$

where $\mathrm{Geg}_\ell(x)$ is a Gegenbauer polynomial,

$$\mathrm{Geg}_\ell(x) = C_\ell^{(d/2-1)}(x)\,, \tag{3.64}$$

and the normalization factor $\mathcal{N}_{d,\ell}$ is given by[12]

$$\mathcal{N}_{d,\ell} = \frac{\ell!}{(-2)^\ell (d/2-1)_\ell}\,. \tag{3.65}$$

By solving Eq. (3.61) one can find conformal blocks for even d. They are expressed in terms of the basic function

$$k_\beta(x) = x^{\beta/2}{}_2F_1\left(\frac{\beta-\Delta_{12}}{2}, \frac{\beta+\Delta_{34}}{2}, \beta; x\right)\,, \tag{3.66}$$

which satisfies

$$\mathcal{D}_x k_\beta(x) = \frac{1}{2}\beta(\beta-2)k_\beta(x)\,, \quad k_\beta(x) \underset{x\to 0}{\sim} x^{\beta/2}\,. \tag{3.67}$$

In the simplest case of $d = 2$, we have $\mathcal{D} = \mathcal{D}_z + \mathcal{D}_{\overline{z}}$, so the conformal blocks factorize. They take the form[13]

$$d = 2: \quad g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\overline{z}) = \frac{1}{(-2)^\ell(1+\delta_{\ell 0})} \times \left(k_{\Delta+\ell}(z)k_{\Delta-\ell}(\overline{z}) + z \leftrightarrow \overline{z}\right)\,. \tag{3.68}$$

[12]Here $(a)_n$ stands for the Pochhammer symbol.

[13]Notice that the 2d global conformal blocks discussed here should be distinguished from the Virasoro conformal blocks.

Results for higher even d can then be found using recursion relations relating blocks in d and $d+2$ dimensions. The important case of $d=4$ reads

$$d=4: \quad g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) = \frac{1}{(-2)^\ell} \times \frac{z\bar{z}}{z-\bar{z}} \left(k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - z \leftrightarrow \bar{z}\right) . \tag{3.69}$$

In odd d, general closed-form solutions of the Casimir equation are so far unavailable. Nevertheless there are very efficient series representations that are sufficient for practical purposes.

Finally, let us mention that conformal blocks have simple transformation properties under the interchange of external operators $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$:

$$\begin{aligned} g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u/v,1/v) &= (-1)^\ell v^{\frac{\Delta_{34}}{2}} g_{\Delta,\ell}^{-\Delta_{12},\Delta_{34}}(u,v) \\ &= (-1)^\ell v^{-\frac{\Delta_{12}}{2}} g_{\Delta,\ell}^{\Delta_{12},-\Delta_{34}}(u,v) . \end{aligned} \tag{3.70}$$

This follows from the symmetry of the OPE under the same interchange. As a check, the explicit expressions in Eqs. (3.68-3.69) satisfy these relations.

Exercise 3.6. *Decompose the 4pt functions $\langle\phi\phi\phi\phi\rangle$ in conformal blocks in $d=4$. Show that all operators appearing in the formula satisfy $\Delta = \ell+2$ and all OPE coefficients squared are positive.*

4 Crossing Symmetry

The main idea of the conformal bootstrap is to constrain CFT data by using the crossing relations for 4pt functions, Fig. 3. Crossing relations are usually analyzed in the conformal frame Eq. (2.19). Consider the 4pt function of scalar operators in this frame and expand it into conformal blocks in the (12)-(34) and in the (32)-(14) OPE channels, referred to as the s- and t-channels.

Since the choice of pairing of the operators is completely arbitrary, the two expansions must give the same final result. The two channels are obtained by interchanging points 1 and 3, which transforms $z \to 1-z$. Taking into account the value of the $\mathbf{K}_4$ prefactor in both channels, and equating the two CPW decompositions, we get the crossing relation:

$$\sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} \frac{g_{\Delta_\mathcal{O},\ell_\mathcal{O}}^{\Delta_{12},\Delta_{34}}(z,\bar{z})}{(z\bar{z})^{\frac{\Delta_1+\Delta_2}{2}}} = \sum_{\mathcal{O}'} \lambda_{32\mathcal{O}'}\lambda_{14\mathcal{O}'} \frac{g_{\Delta_{\mathcal{O}'},\ell_{\mathcal{O}'}}^{\Delta_{32},\Delta_{14}}(1-z,1-\bar{z})}{((1-z)(1-\bar{z}))^{\frac{\Delta_3+\Delta_2}{2}}} \tag{4.1}$$

Here the sums run over the operators $\mathcal{O}$ and $\mathcal{O}'$ which appear in the OPE in the two channels.

One frequently occurring special case is a 4pt function of identical scalars $\langle\sigma\sigma\sigma\sigma\rangle$. Then the crossing relation simplifies because $\mathcal{O}=\mathcal{O}'$ and also because we get squares of the OPE coefficients $\lambda_{\sigma\sigma\mathcal{O}}$. It is customary to write it as

$$\sum_{\mathcal{O}} \lambda_{\sigma\sigma\mathcal{O}}^2 F_{\Delta_\mathcal{O},\ell_\mathcal{O}}^{\Delta_\sigma}(z,\bar{z}) = 0 , \tag{4.2}$$

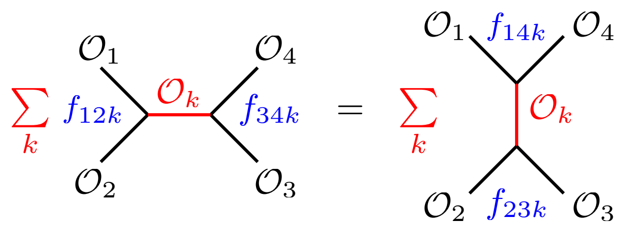

Figure 3: Crossing relation for the 4pt function $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$.

where

$$F^{\Delta_\sigma}_{\Delta,\ell}(z,\bar{z}) = ((1-z)(1-\bar{z}))^{\Delta_\sigma} g^{0,0}_{\Delta,\ell}(z,\bar{z}) - (z\bar{z})^{\Delta_\sigma} g^{0,0}_{\Delta,\ell}(1-z,1-\bar{z}). \tag{4.3}$$

Among the operators $\mathcal{O}$ which appear in (4.2), a special role is played by the identity operator and (in local CFTs) by the stress tensor, because these are two operators of known dimension whose OPE coefficients are nonzero. In particular the identity operator appears with the coefficient $\lambda_{\sigma\sigma\mathbb{1}} = 1$. By studying the $z \to 0$ limit of the crossing relation, it's easy to show analytically that there should be infinitely many further operators with nonzero $\lambda_{\sigma\sigma\mathcal{O}}$ [**?**]. We will see later on what can be learned about these operators using numerical methods.

Going back to the general case (4.1), it is similarly convenient to rewrite it as follows [**?**]. We introduce the functions

$$F^{ij,kl}_{\pm,\Delta,\ell}(z,\bar{z}) = ((1-z)(1-\bar{z}))^{\frac{\Delta_k+\Delta_j}{2}} g^{\Delta_{ij},\Delta_{kl}}_{\Delta,\ell}(z,\bar{z}) \pm (z\bar{z})^{\frac{\Delta_k+\Delta_j}{2}} g^{\Delta_{ij},\Delta_{kl}}_{\Delta,\ell}(1-z,1-\bar{z}), \tag{4.4}$$

which are symmetric/antisymmetric under $z \to 1-z$, $\bar{z} \to 1-\bar{z}$. We then take the sums and differences of (4.1) with the same equation with $z, \bar{z}$ replaced by $1-z, 1-\bar{z}$. Then (4.1) is equivalent to the pair of equations:

$$\sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} F^{12,34}_{\mp,\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(z,\bar{z}) \pm \sum_{\mathcal{O}'} \lambda_{32\mathcal{O}'}\lambda_{14\mathcal{O}'} F^{32,14}_{\mp,\Delta_{\mathcal{O}'},\ell_{\mathcal{O}'}}(z,\bar{z}) = 0\,. \tag{4.5}$$

If all operators are equal, the lower sign case is trivial, and the upper sign reduces to the single correlator crossing relation (4.2).

Crossing relations can be imposed at any point $z, \bar{z}$ where both the s- and t-channels converge. From the discussion in Sec. 3.4, this is the plane of all complex z minus cuts along $(1, +\infty)$ where the s-channel diverges and $(-\infty, 0)$ where the t-channel diverges. As we will see in Sec. **??**, the standard choice in numerical studies is to impose crossing in a Taylor expansion around the point $z = \bar{z} = 1/2$, which is well inside this region.

There is also a third u-channel OPE (13)-(24). The u-channel is typically not considered in the numerical bootstrap, because it is not convergent at $z = \bar{z} = 1/2$.[14] For 4 identical external scalars, the u-channel is automatically satisfied if the s-t channel crossing relation

[14] Although it can be considered when crossing relations are analyzed around another point, e.g. $u = v = 1$ [**?**].

holds [?]. For nonidentical external operators, the u-channel is important. To impose the u-channel crossing relation, one changes the conformal frame by interchanging the positions of operators 1 and 2 [?]. The u-channel in the original frame becomes the t-channel in the new frame, and the s-u crossing can be imposed at $z = \bar{z} = 1/2$. The s-channel CPW decomposition in the new frame only differs by signs of all odd-spin terms because of (??).

In the case when CFT has a global symmetry G, crossing relations were formalized in [?]. Consider a 4pt function of scalar operators transforming in representations π_i. The exchanged operators $\mathcal{O}_\pi$ then transform in representations π appearing in the tensor product decompositions of $\pi_i \otimes \pi_j$. Each term in the s- and t-channel CPW decompositions comes multiplied with a tensor structure obtained by contracting two 3pt G-invariant tensors, as described in Sec. ??. We can represent it by a vector $\vec{V}_\pi$ in the space of 4pt G-invariant tensors $(\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4)^G$. (Anti)symmetrizing under $z \to 1 - z$, $\bar{z} \to 1 - \bar{z}$, the crossing relation takes form (4.5) with every term multiplied by the corresponding vector $\vec{V}_\pi$. It is thus a constraint in the space of vector functions.

Exercise 4.1. *Obtain the crossing equations of 4pt functions $\langle \phi_i \phi_j \phi_k \phi_l \rangle$ and $\langle \phi_i \phi^{\dagger \bar{\jmath}} \phi_k \phi^{\dagger \bar{l}} \rangle$ for ϕ_i a fundamental of $SO(N)$ or $SU(N)$*

5 Conformal Bootstrap